The Pragmatism of Hilbert's Programme*

Volker Peckhaus
Institut für Philosophie
der Universität Erlangen-Nürnberg
Bismarckstr. 1, D – 91054 Erlangen
E-mail: vrpeckha@phil.uni-erlangen.de

1 Introduction

On 17 June 1899 the Göttingen society took part in the festive unveiling of the Gauss-Weber monument in memory of the great mathematical and physical tradition of the University of Göttingen. On the occasion of this event a commemorative volume was published containing two contributions: Emil Wiechert's paper on the foundation of electrodynamics (Wiechert *1899*) and David Hilbert's paper on the foundations of geometry (Hilbert *1899*).

Hilbert's paper became a mathematical bestseller, Hilbert himself one of the leading figures in the world of mathematics. This is best illustrated by the fact that Hilbert was honored to speak at the Second Congress of Mathematicians that took place in Paris in August 1900 on the occasion of the centenary world's fair at that place. Hilbert's lecture "Mathematical Problems" (Hilbert *1900*) was intended to set and at least partly succeeded in setting the mathematical agenda for the new century. Hilbert was able to capture "the imagination of the mathematical world with his list of problems for the twentieth century," as Constance Reid wrote (*1970*, 84). "His rapidly growing fame," she continues, "promised that a mathematician could make his reputation for himself by solving one of the Paris problems" (ibid.).

In his *Foundations of Geometry*, published one year earlier, Hilbert introduced paradigmatically what can be called a formalist foundation of a branch of mathematics. This position in the philosophy of mathematics is

*Lecture delivered at the "History of Logic" conference at the University of Helsinki in June 2000; at the symposium "Cien años de los 'Problemas matemáticos' de David Hilbert" in Buenos Aires in August 2000; and at the Fourth International Congress of the Society of Analytical Philosophy in Bielefeld in September 2000. To be published in the *Synthese Library.*

usually opposed to Logicism and Intuitionism. It is understood as based on an axiomatical deductive method, organized like Euclid's *Elements*, but independent from any intuitive base and sufficiently justified by meta-axiomatical investigations on completeness, independence and consistency.

In the following presentation it will be shown that this formalistic side of Hilbert's approach is accompanied by a certain pragmatism that is compatible with a philosophical, or, so to say, external foundation of mathematics. One can even say that Hilbert's foundational programme can be seen as a reconciliation of Pragmatism and Apriorism. This interpretation is elaborated by discussing two recent positions in the philosophy of mathematics which are or can be related to Hilbert's axiomatical programme and his formalism. In a first step it is argued that the pragmatism of Hilbert's axiomatic contradicts the opinion that Hilbert style axiomatical systems are closed systems, a reproach recently posed by Carlo Cellucci (Cellucci *1993*, *1996*, *1998*, *2000*). In the second section the question is discussed whether Hilbert's pragmatism in foundational issues comes close to an a-philosophical "naturalism in mathematics" as suggested by Penelope Maddy in her recent book (Maddy *1997*). The answer is "no", because for Hilbert philosophy had its specific tasks in the general project to found mathematics. This is illuminated in the concluding section giving further evidence for Hilbert's foundational apriorism.

2 Hilbert's Axiomatics as Open System

In his *Foundations of Geometry* Hilbert does not really reflect on mathematical methodology. He actually *gives* a foundation of Euclidean geometry which he calls "a new attempt" for establishing a simple and complete system of independent axioms that allows to deduce the most important geometrical theorems in such a way that the significance of the different groups of axioms can be recognized and the consequences of certain axioms become clear (Hilbert *1899*, 4). The basic objects of his system of axioms are "thought-things", i. e., products of human thought (ibid., 4). Geometry now becomes a speculative discipline, its relation to intuition becomes irrelevant, or, as Hans Freudenthal took it, the connection between reality and geometry is cut (Freudenthal *1957*, 111). The non-intuitive character of the new geometry is one of the main differences to the Euclidean model. Geometry becomes a branch of pure mathematics. Additionally, axioms are no longer evident truths. According to Freudenthal, it does not even make sense to ask for their truth (ibid.). The Euclidean justification of the axioms with the help of intuition and evidence is replaced by three meta-axiomatical conditions: the axiomatical system has to be consistent, the axioms have to be independent from one another and the axiomatical system has to be complete. This can be interpreted as a dissolution of geometry from philosophy, or at least from some philosophical aspects being usually connected with mathematics. As an example Paul Bernays might be quoted who wrote in 1922 (Bernays *1922*; quote according to Mancosu *1998*, 192):

> The important thing [...] about Hilbert's "Foundations of Geometry" was that here, from the beginning and for the first time, in the laying down of the axiom system, the separation of the mathematical and logical [spheres] from the spatial-intuitive [sphere], and with it from the epistemological foundation of geometry, was completely carried out and expressed with complete rigor.

As mentioned earlier, Hilbert's *Foundations of Geometry* is not an essay on method, but the result of the application of a method, the axiomatical method. Hilbert started, so to say, with the model, providing the theory in later contributions. The axiomatical method has been interpreted as an attempt of providing a final or absolute foundation of mathematics. Hilbert's comments in the problems lecture seem to support this interpretation, especially his elucidations concerning the second problem on "The Compatibility of the Arithmetical Axioms" (Mary Winston Newson's translation used). When being engaged in investigating the foundations of science, Hilbert says, we have to "set up a system of axioms which contains an exact and complete description of the relations subsisting between the elementary ideas of that science" (Hilbert *1902b*, 447). Hilbert renders as the most important question with regard to the axioms: "To prove that they are not contradictory, that is, that a finite number of logical steps based upon them can never lead to contradictory results" (ibid.). In this programme the consistency proof for the axioms of arithmetic has an outstanding position: "the proof of the compatibility of the axioms [of arithmetic] is at the same time the proof of the mathematical existence of the complete system of real numbers or of the continuum" (448). Such proof would furthermore end the preliminary state of all relative consistency proofs like Hilbert's own proof of the consistency of geometry in his *Foundations of Geometry*, where he had shown that geometry is consistent if arithmetic is (cf. ibid., 447). A consistency proof for the axioms of arithmetic would provide a confirmation of what Hilbert called a "conviction" or even an "axiom" in the problems lecture, what others had labeled "Hilbert's dogma" (ibid., 444)

> that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts

For Hilbert, there is no *ignorabimus* in mathematics (ibid., 445), against Emil du Bois-Reymond's conviction "ignoramus et ignorabimus"—we are ignorant and we remain ignorant.

Carlo Cellucci has called axiomatical systems resulting from this standpoint "closed systems". In the light of Gödel's results such an attempt has to fail necessarily, says Cellucci (*1993*, cf. Cellucci 1998, and Gillies' review *1999*). In Cellucci's reconstruction, the 19th century founders of mathematical logic had identified the axiomatical method, i. e., the axiomatic-deductive method, with the mathematical method. And investigating this mathematical method has been the task of mathematical logic. Cellucci criticizes three basic assumptions of today's mathematical logic (*1993*, 211):

> 1. The mathematical method is to be identified with the axiomatical method.
> 2. Although by Gödel's result no single formal system can represent the whole of mathematics, there are formal systems that are adequate for representing current mathematical practice.
> 3. In view of this, the notion of formal system as a closed system is adequate for mathematics and is unaffected by Gödel's result.

Cellucci objects that "the concept of formal system as a closed system is inadequate for mathematics" (ibid., 212). His alternative suggestion is based on the demand (ibid.),

> that each formal system for any branch of mathematics containing number theory must admit proper extensions, and hence the choice of any particular formal system would be intrinsically provisional, subject to an eventual need to go beyond it. Therefore the concept of formal system as a closed system is incapable of representing the mathematical process.

Cellucci suggests that mathematics should be errected as an "open system". Open mathematics has to be based on a new paradigmatic logic, a "computational logic" that has some features in common with the programming language PROLOG. In PROLOG axioms and inference rules can be changed during a proof with the help of the predicates "assert" and "retract".

Cellucci is right in emphasizing the importance of analytical-regressive methods against the usual preference for synthetical-progressive methods or ways of presentation. He is right in demanding that the mathematical process, i. e., the process of producing mathematics, has to be considered in logic and in the philosophy of mathematics. He is wrong, however, in his historical evaluation of Hilbert's foundational approach. A closer look on Hilbert's own writings reveals rather the mythical character of the historical picture he draws for motivating his suggestion. First of all, we have to ask: what is the axiomatical method according to Hilbert? Its role within the axiomatical programme is neatly expressed in Hilbert's lecture on "Axiomatic Thought" delivered before the Swiss Mathematical Society in Zurich in September 1917 (Hilbert *1918*). There Hilbert compares mathematical research with the reconstruction and enlargement of a building (Hilbert *1918*, quote according to Ewald [ed.] *1996*, 1109):

> The procedure of the axiomatic method, as is expressed here, amounts to a deepening of the foundations of the individual domains of knowledge—a deepening that is necessary for every edifice that one wishes to expand and to build higher while preserving its stability.

This quotation shows Hilbert's foundational pragmatism. The axiomatical method is a tool of supplementary and supporting character which aims at keeping mathematics running. As early as 1902/1903 Hilbert defined the axiomatical method as a procedure of finding for a given proposition the premises from which it follows. He wrote (*1902/03*, 50):

> I understand under the *axiomatical* exploration of a mathematical truth [or theorem] an investigation which does not aim at finding new or more general theorems being connected with this truth, but to determine the position of this theorem within the system of known truths in such a way that it can be clearly said which conditions are necessary and sufficient for giving a foundation of this truth.

Hilbert thus aims at using the axiomatical method as an architectural procedure which determines the hierarchical order of a network of statements, based on the relations between conditions and consequences. The axiomatical method can thus be identified with what Pappus defined as the analytical method, or with the regressive or critical method of traditional philosophy (cf. Pappus *1986*, 82). Its objective in Hilbert's axiomatical programme is "deepening of foundations" in those branches of mathematics where the foundations had been questioned. This is a dynamical process in the sense that the foundations are deepened as far as necessary for providing the consistency of the respective branch of mathematics. It does not aim at a "final" or "absolute" foundation. Hilbert illustrates this pragmatic aspect in a discussion of the set-theoretic paradoxes which can be found in a lecture course delivered in Göttingen in 1905 (Hilbert *1905b*, 122):

> It had, indeed, been usual practice in the historical development of science that we began cultivating a discipline without many scruples, pressing onwards as far as possible, that we thereby, however, run into difficulties (often only after a long time) that forced us to turn back and reflect on the foundations of the discipline. The house of knowledge is not erected like a dwelling where the foundation is first

> well laid-out before the erection of the living quarters begins. Science prefers obtaining comfortable rooms as quickly as possible in which it can rule, and only subsequently, when it becomes clear that, here and there, the loosely joined foundations are unable to support the completion of the rooms, science proceeds in propping up and securing them. This is no shortcoming but rather a correct and healthy development.

These quotations show that according to Hilbert's background philosophy axiomatical systems are far from being closed systems.

3 Naturalism

Given this reconstruction, Hilbert's foundational position seems to be a perfect example of what Penelope Maddy had called "naturalism in mathematics" (Maddy *1997*). In her recent book she rejects her earlier *Realism in Mathematics* (Maddy *1990*) and, together with the ontological model suggested there, all other "philosophical" approaches. Her new position is modeled on Quine's who held that science is "not answerable to any supra-scientific tribunal, and not in need for any justification beyond observation and the hypothetico-deductive method" (Quine *1975*, 72). In analogy Maddy propagates the mathematical naturalist's position "that mathematics is not answerable to any extra-mathematical tribunal and not in need of any justification beyond proof and the axiomatic method" (Maddy *1997*, 184). Given the case of set theory where almost all methodological questions are settled in practice, but most of the corresponding philosophical debates have not been finished yet (191), she denies the right of philosophy to disturb successful mathematical practice. She even goes so far as to propose some sort of "pleasure theory of truth", when suggesting (163):

> If it is legitimate, in the set theoretic case, to argue from "this theory has properties we like" to "this theory is true", if this is not just a form of wishful thinking, then it appears that we are free to extend our set theory in any way that suits our need [...].

The question now arises, who gives the justification for techniques of proof or for the axiomatical method itself? Is it enough to reduce philosophical argument to the question whether certain proof methods are settled in practice or not? If so, philosophy of mathematics is reduced to a description of settled practice. And if naturalism in mathematics is "largely non-philosophical", as Maddy concedes (233), it remains open whether her "naturalistic philosopher" (200–205) is a philosopher at all, or some sort of judge comparing new elements of practice with their settled counterparts, having at hands only one criterion: usefulness in present-day practice.

Maddy anticipates arguments like this. In her discussion of Quine's scientific naturalism she writes (181):

> Opponents of naturalism sometimes complain that the naturalistic philosopher is reduced to recording the pronouncements of scientists, that such philosophy has no critical function, that it is reduced to mere sociology of science.

Maddy holds that this is not true (ibid.):

> Natural science [and one could add, mathematics as well] is a self-critical enterprise that develops and debates its own methodological norms. The naturalistic philosopher is free to join in this part of ongoing science, like anyone else, except that she cannot expect to use any peculiarly philosophical methods. The only available methods are the scientific ones; for he naturalist, the evaluation and assessment of scientific method must take place within science, using those very methods themselves.

Maddy argues as if science and mathematics were closed in respect to the methods they use. She argues as if it had once and for all been determined what scientific and mathematical methods are, or that any development of methodology has to come from inside science and mathematics. The philosopher can join the discussion, but he or she has nothing to say, not even that the demand to justify methods using the methods themselves is clearly circular and therefore not a good advice. The philosopher is external, the representative of a different branch of knowledge.

Maddy is writing against foundational positions like Logicism, Formalism, and Intuitionism. Their representatives are the "philosophers" she wants to get rid of. The case of philosophy is, however, much more complicated. Doing philosophy is an activity which is not bound to any institutional framework or even to the professional status of the one who does it. If a mathematician or scientist is self-critically developing and debating his or her own methodological norms, he or she is doing philosophy. This is what Maddy says, but there is no good reason to maintain that the professional philosopher, the logician or the proof theorist have no say in this "self-critical enterprise". There is no strict border between the philosophizing mathematician or scientist and the professional philosopher of mathematics or science. One should not force the self-critical mathematician or the scientist to invent the wheel again and again. The philosopher coming from outside mathematics or science might be capable to give them orientation for solving foundational problems. He might open the scientist's eyes for problems and ideas which are relevant for his problem, but not in the focus of his usual research.

Let us return to Hilbert. Hilbert was a naturalist insofar, as his foundational attempts aimed at securing traditional mathematics in cases where foundational problems and anomalies had occurred. But he knew that Formalism was a mathematical programme which had to be supplemented by a philosophical foundation. In his early writings he preferred a logicistic foundation based on a Kantian epistemology, and he knew that he, as a professional mathematician, was not competent enough to handle this philosophical task. Therefore he looked for professional philosophical assistance, supporting in Göttingen, e. g., the Neo-Friesian philosopher Leonard Nelson and the Phenomenologist Edmund Husserl (cf. Peckhaus *1990*, 196–224). Later he developed the meta-mathematical programme, i. e., proof theory based on strict finitism which is itself a real philosophical programme (cf. Sieg *1999*). Again he was assisted by a trained philosopher, Paul Bernays, who can be called the architect of proof theory.

One of the reasons for Hilbert's openness for philosophical questions and for collaboration with philosophers was his opposition to a strict distinction of branches of knowledge. He was an adherent of the Cartesian and Leibnizian idea of a *mathesis univeralis*, i. e., a general science underlying all branches of knowledge and therefore transcending the borders of professional mathematics.

To sum up: Hilbert's methodological pragmatism is no naturalism because Hilbert supplements the formulation of a foundational system being sufficient for mathematical purposes with a philosophical justification of this system. The foundational enterprise is a dynamical procedure. If there is any idea of final foundation (*Letztbegründung*) in the background, it has only a heuristic function, i. e., it served as a regulative idea in the Kantian sense. The axiomatical method is understood as a general method, not restricted to proper mathematics. It holds for all domains of knowledge in which structurable arguments occur. If this is granted, it follows that reflection on this kind of "axiomatical method" is not restricted to the competence of mathematicians. The basic problem of all applications of the mathematical method is to justify the choice of a set of certain statements as starting points of deductions. This is clearly a philosophical problem which cannot be solved by simply referring to the success of a certain proposal as is done in a Maddy style Naturalism

4 Hilbert's Programme and Philosophy

The last section deals with some aspects of Hilbert's apriorism, especially by connecting it to Leonard Nelson's attempt of formulating a "critical mathematics".

Several interpreters deny a connection between Hilbert's pragmatic formalism and philosophy. On the contrary, they consider Formalism as a successful attempt to extinguish any philosophical influence in mathematics. Herbert Mehrtens might serve as an example who interpreted Hilbert's introduction of the objects of geometry in his foundation of geometry as "creation" that appears to be "creation from nothing". The mathematician is the autonomous master of a world created by himself (Mehrtens *1990*, 124). He separates his work sharply from philosophy. He does not accept any normative influence of philosophy (ibid., 130*f*.). And indeed, Hilbert's formalism aiming at an internal justification of mathematical sets of propositions cannot really be regarded as a philosophical foundational positions.

Nevertheless, this picture is false. In Hilbert's foundational programme, philosophy comes in at several places. In the Hilbert circle, axiomatical systems were regarded as hypothetico-deductive systems in a naive sense. I. e., they are deductive systems, and the deductions start from statements regarded as hypotheses. Paul Bernays, e. g., stressed (*1930/31*, quoted in *1976*, 19–20) that the relation between axioms and theorems is purely hypothetical. "If it is the case what the axioms assert, the theorems are valid." Thus, the truth of mathematical theorems is a relative truth, relative to the truth of the axioms themselves. Mathematics abstracts from the truth of the axioms, and it is therefore possible to abstract from any intuitive (*anschaulichen*) content of the theory (ibid.). However, Bernays' position does not answer the question what epistemological status the axioms have. The answer is simply left open. It is a philosophical task to clarify this status, and therefore the question marks an anchorpoint of philosophy in the business of founding mathematics.

4.1 Hilbert's Axiom of Reasoning

Even if one accepts that formalistic axiomatical mathematics is unaffected of intuition, it does not follow that mathematics is entirely free from any philosophical element. Hilbert connected his foundational efforts with strong idealistic assumptions. The "thought things" as basic objects in the *Foundations of Geometry* are one example. Another example are Hilbert's critical considerations on the conditions being necessary for producing any mathematics at all. These conditions can be understood in the sense of the old postulates which give the conditions for the possibility to act in a certain way. A postulate of this kind is Hilbert's "axiom of reasoning" or "axiom of the existence of mind" presented in the lecture course *Logische Principien des mathematischen Denkens* (*1905a*, 219):

> I have the ability to think *things*, and to designate them by simple signs $(a, b, \ldots X, Y, \ldots)$ in such a completely characteristic way that I

> can always recognize them again without doubt. My thinking operates with these designated things in certain ways, according to certain laws, and I am able to recognize these laws through self-observation, and to describe them perfectly.

Hilbert denotes this principle as "axiom", although it sets the conditions of mathematical operations and although it claims to be materially true and not only generally valid. In a marginal note to Ernst Hellinger's lecture notes Hilbert calls this "axiom" the "philosophers' *a priori*". This shows that he wanted to see the responsibility for a justification of this principle in the hands of philosophers. He thus granted philosophy and important role within the enterprise to found mathematics.

4.2 Ernst Schröder's "One and Only Axiom"

Hilbert did not stand alone with the attempt of founding mathematics extra-mathematically with the help of axioms of that kind. As a further example the outstanding German representative of the algebra of logic Ernst Schröder (1841–1902) can be mentioned. In his *Lehrbuch der Arithmetik und Algebra*, a textbook published in *1873*, Schröder gave a foundation of a "formal algebra". He created this algebra according to the model of the general doctrines of forms suggested by Hermann Günther Graßmann and Hermann Hankel. The formal algebra understood as a general theory of connections included the algebraical structure, i.e., the algebra of logic (cf. Peckhaus *1996*, *1997*, ch. 6).

Schröder's foundation of arithmetic had its effect. As late as 1898 it was adopted by Hermann Schubert in his contribution on "Grundlagen der Arithmetik" for the *Encyklopädie der mathematischen Wissenschaften* (Schubert *1898*) which provoked Gottlob Frege's serious polemics in the pamphlet *Ueber die Zahlen des Herrn H. Schubert* (Frege *1899*), which were in fact the numbers of Mr. Schröder.

Schröder's thoughts about formal algebra can be found in the "formal" part of the textbook. According to its intended function as a textbook for mathematics teachers at grammar schools, Schröder started, however, with an introductory "real" part in which he treated the theory of natural numbers. Schröder intended to build up arithmetic as far as possible on conventional stipulations. For this program he thought to get by with one single axiom. It should be noted that Schröder used the term "axiom" in its traditional sense, as a self-evident proposition. Strictly speaking Schröder's "one and only" axiom is a classical postulate on the behaviour of objects. It is a presuppositon for Schröder's definition of the natural number as the sum of units. "Units" represent countable objects, symbolized by strokes (*1873*, 5). Hence, numbers are characterized by a counting procedure that is only possible under the precondition that a sign, once set, will persist. This precondition is for Schröder "an axiom for every deductive science at all." The "one and only" axiom for arithmetic is the "axiom of the inherence of signs". It gives us the certainty, says Schröder (16–7),

> that in all of our derivations and conclusions the signs will persist in our memory—and even more constantly on the paper. [...] Without this principle received by induction and generalization from a very rich experience, indeed every induction would be illusionary, because a deduction begins just at the moment when—after having suffiently clothed the properties of the objects into signs—the investigation of the objects is replaced by the investigaton of their signs.

It is due to the empirical evidence of this axiom that Schröder calls it an assumption "which has only a greater or lower degree of probability, depending on the calculating person's power of recollection or of the properties of the used material". This probability would be very low, he continues, "if one would write with a volatile or secret ink." Despite this restriction, Schröder thought that he could base our conviction of the absolute certainty of mathematical truths on the reliability of this axiom in practise (17).

Of course, Schröder's considerations are circular. His axiom determines the behaviour of material objects which is a precondition for sucessfull operations with signs in mathematics, and at the same time this behaviour is obtained phenomenologically by induction over this behaviour. So it is not astonishing that he had to receive serious criticism for his "one and only" axiom from Gottlob Frege (*1884*, VIII) and Benno Kerry (*1890*, 333–336), and still in 1927 it was the reason for polemics of the Göttingen neo-Friesian philosopher Leonard Nelson. This last polemics brings us back to Hilbert.

4.3 Nelson's Criticism

Leonard Nelson was the founder of the *Neue Fries'sche Schule* (cf. Peckhaus *1990*, ch. 5). In continuation of the philosophical systems of Immanuel Kant and Jakob Friedrich Fries, he propagated a "critical philosophy" centering around the "regressive method", i. e., a method of discovering those philosophical principles which are the foundation of our everyday experiences.

"If we pick out of the experiences of life", he writes (*1904*, 4–5),

> such decisions and judgements concerning which a consensus exists, we can dismember them, and so, by a regressive method, trace the common philosophical principles that were presupposed and applied in reaching these decisions and judgements.

A certain parallelism with Hilbert's axiomatic method of discovering mathematical axioms is evident, and so it is obvious that Nelsons extended his interests to mathematics. Among his philosophical aims was a "critical deduction of the axioms of mathematics" (*1904*, 37) within the framework of a "critical mathematics", developed together with his friend, the geometer Gerhard Hessenberg (1874–1925, cf. Hessenberg *1904*). Critical mathematics or philosophy of mathematics comprises beside the mathematical task of examining axiomatic systems, a second, philosophical task of investigating the apriori conditions for the validity of the mathematical axioms and on this way for mathematical truths as such (Nelson *1906*, 149). For mastering these tasks it is important "to distinguish within a mathematical discipline sharply between objects which can be proved logically, and those that are intuitive preconditions for such a proof" (*1928*, 109). The last quote comes from a lecture entitled "Critical Philosophy and Mathematical Axiomatic", delivered by Nelson on the occasion of the 56th meeting of philologists and teachers at Göttingen in September 1927, the year of his death. The lecture was published posthumously in 1928, accompanied by a memorial address written by David Hilbert. At the time of the lecture Nelson stood under the impression of attempts in the Hilbert circle of proving the consistency of the arithmetical axioms then believed to be successfully finished. Nelson emphasized as a task of critical mathematics to deduce the mathematical axioms to immediate, i. e., not logically derived and not empirical knowledge underlying the mathematical axioms, based on pure intuition of space and time (*1928*, 112). In the discussion period Nelson's lecture was sharply criticized by Richard Courant and Paul Bernays. Because of this criticism Nelson felt urged to an unusual move. He published a very polemical "response" almost as long as the lecture itself. One should have in mind, that both, Courant and Bernays, had been among the early followers of the neo-Friesian movement. According to Nelson's summaries of Courant's and Bernays' arguments they had doubted in a formalistic sense that mathematics provides any knowledge [*Erkenntnis*]. Courant suggested to restrict the usual notions of truth and validity to metamathematics, whereas Bernays went further holding that mathematical axioms are not connected with any claim of being knowledge at all. Bernays said that this follows from the demand to prove the consistency of axiomatical systems. If one succeeds in reducing the axioms to immediate and true knowledge, these axioms are themselves immediate and true knowledge, i. e., a proof of consistency would be superfluous. In such a kind of mathematics operating with consistency proofs the doctrine of founding axioms on pure intuition is untenable.

Not all of Nelson's specific rejoinders can be discussed, only those passages will be mentioned where he refers to Schröder's axiom (*1928*, 140) in order to show the absurdity of Bernays' criticism. If pure intuition of space as a source of knowledge for mathematics is disavowed in metamathematics one has to consider, in consequence, empirical intuition. The signs with which the metamathematician operates or rather experiments

> are therefore chalk strokes on the blackboard or traces of ink on the paper which can be perceived sensually. In order to find inferences which are generally valid metamathematics needs, however, certain preconditions concerning the steadiness of its signs; and these preconditions have to be themselves apodictically valid. If the object of metamathematics are the chalk strokes on the blackboard, then mathematics needs an apodictically certain axiom, claiming that these chalk strokes are constant and that it is possible to produce them at any place of the blackboard. And this axiom as an apodictical judgement would be based on knowledge *a priori*. Mind you: knowledge *a priori* of the everlastingness of chalk! The one who believes that he can dispense with the foundation of mathematics on pure intuition of space has to dare instead a knowledge *a priori* on the fate of blackboard and chalk.

Such mathematical empiricism would turn everything upside down: "After the loss of pure mathematics, which gave us the conditions for sensible experimenting, we find us in control of a *metaphysics of chalk* scorning any experiment." The attempt of "finally founding mathematics on conditions concerning the nature of the writing material", he continues, had been seriously undertaken by no less a person than the "well-known founder of the algebra der logic", as Nelson incorrectly called Ernst Schröder.

Does Nelson's criticism concern Hilbert's axiom as well? Nelson obviously knew the axiom since he attended the respective lecture in the summer term of 1905. At a glance the two axioms of Hilbert and Schröder seem to be quite similar. While Schröder speaks of certainty won by induction that chalk strokes representing units are steady, Hilbert postulates the capacity to think things and symbolize them by definitely recognizable signs, e. g., by chalk strokes. Both axioms are similar in their demand that the symbols used should be definitely recognizable. A difference can be seen in the fact that Schröder formulates his axiom in respect to what is recognized, i. e., ontologically, whereas Hilbert's axiom is focused on epistemological preconditions. If Hilbert had deduced the demanded mental capacity from factual acts of reasoning or from facts of the physiology of the brain, Nelson's argument against empirical mathematics could be directed against Hilbert's axiom as well. Anyway, for both axioms we can say, however, that they concern the conditions of formal axiomatics, i. e., they are not part of it. This necessary separation is veiled by naming such conditions "axioms".

Hilbert's and Schröder's attempts of founding mathematics extra-mathematically had admittedly their defects. It should be noted, however, that both authors were conscious about the problem to found foundations themselves. Hilbert knew that Formalism as characterized by formal axiomatics provides only a relative foundation of mathematics, relative insofar as it sets only the proper domain of the mathematicians' work on secure, i. e., consistent foundations. This domain is quite in the air, it is surrounded by non-mathematical contexts which nevertheless influence mathematics. According to the state of mathematical knowledge the foundations of mathematics are fixed only insofar as it is necessary for mathematical practice. This is expressed in Hilbert's metaphor of deepening the foundations as a task of foundational studies (Hilbert *1918*, 417). In this opinion, that what can be called an "absolute" foundation of mathematical knowledge does not concern the mathematician. It is the task of philosophers, but part of the big project of a universal mathematics, in which both, mathematicians and philosophers, are involved.

References

BERNAYS, Paul *1922* "Die Bedeutung Hilberts für die Philosophie der Mathematik," *Die Naturwissenschaften* **31**, 93–99; Engl. translation "Hilbert's Significance for the Philosophy of Mathematics," in Mancosu *1998*, 189–197.

— *1930/31* "Die Philosophie der Mathematik und die Hilbertsche Beweistheorie," *Blätter für Deutsche Philosophie* **4**, 326–367; again in Bernays *1976*, 17–61.

— *1976 Abhandlungen zur Philosophie der Mathematik*, Darmstadt: Wissenschaftliche Buchgesellschaft.

CELLUCCI, Carlo *1993* "From Closed to Open Systems," in *Philosophie der Mathematik. Akten des 15. Internationalen Wittgenstein-Symposiums I. 16. bis 23. August 1992. Kirchberg am Wechsel (Österreich)*, ed. Johannes Czermak, Wien: Verlag Hölder-Pichler-Tempsky, 206–220.

— *1996* "Mathematical Logic: What Has it Done for the Philosophy of Mathematics," in Piergiorgio Odifreddi (ed.), *Kreiseliana. About and Around Georg Kreisel*, Wellesley, MA: A K Peters 365–388.

— *1998 Le Ragioni della Logica*, Rome: Laterza (*Biblioteca di Cultura Moderna*; 1136).

— *2000* "The Growth of Mathematical Knowledge: An Open World View," in *The Growth of Mathematical Knowledge*, ed. Emily Grosholz/Herbert Breger, Dordrecht/Boston/London: Kluwer (*Synthese Library*, vol. 289), 153–177.

EWALD, William (ed.) *1996 From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, vol. II, Oxford: Clarendon Press.

FREGE, Gottlob *1884 Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, Breslau: Wilhelm Koebner; critical edition: Frege *1986*.

— *1899 Ueber die Zahlen des Herrn H. Schubert*, Jena: Hermann Pohle; again in Frege *1967*, 240–261, reprinted in Frege *1999*.

— *1967 Kleine Schriften*, ed. Ignacio Angelelli, Darmstadt: Wissenschaftliche Buchgesellschaft; Hildesheim: Olms ²1990.

— *1986 Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl. Centenarausgabe*, with supplementing texts critically ed. by Christian Thiel, Hamburg: Felix Meiner.

— *1999 Zwei Schriften zur Arithmetik. Function und Begriff. Über die Zahlen des Herrn H. Schubert*, ed. Wolfgang Kienzler, Hildesheim: Olms.

Freudenthal, Hans *1957* "Zur Geschichte der Grundlagen der Geometrie. Zugleich eine Besprechung der 8. Aufl. von Hilberts 'Grundlagen der Geometrie'," *Nieuw Archief voor Wiskunde* (4) **5**, 105–142.

Gillies, Donald *1999* Review of Cellucci *1998*, *Philosophia Mathematica* ser. III, **7**, 213–222.

Hessenberg, Gerhard *1904a* "Über die kritische Mathematik," *Sitzungsberichte der Berliner Mathematischen Gesellschaft* **3**, 21–28, 20th convention, 25 November 1903, appendix to *Archiv der Mathematik und Physik* (3) **7**.

Hilbert, David *1899* "Grundlagen der Geometrie," in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*, ed. by the Fest-Comitee, Leipzig, 1–92, 14th ed. Hilbert *1999*. English translation Hilbert *1902a*.

— *1900* "Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900," *Nachrichten von der königl. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse aus dem Jahre 1900*, 253–297. Engl. translation Hilbert *1902b*.

— *1902a The Foundations of Geometry* [translation by E. J. Townsend], Chicago: Open Court.

— *1902b* "Mathematical Problems. Lecture Delivered Before the International Congress of Mathematicians at Paris in 1900," translated by Mary Winston Newson, *Bulletin of the American Mathematical Society* **8**, 437–479.

— *1902/03* "Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck," *Proceedings of the London Mathematical Society* **35**, 50–68.

— *1905a Logische Principien des mathematischen Denkens*, lecture course, summer term 1905, lecture notes by Ernst Hellinger (library of the Dept. of Mathematics, University of Göttingen).

— *1905b Logische Principien des mathematischen Denkens*, lecture course, summer term 1905, lecture notes by M. Born, Staats- und Universitätsbibliothek Göttingen, Cod. Ms. D. Hilbert 558a.

— *1918* "Axiomatisches Denken," *Mathematische Annalen* **78**, 405–415; Engl. translation under the title "Axiomatic Thought" in Ewald (ed.) *1996*, 1105–1115.

— *1999 Grundlagen der Geometrie. Mit Supplementen von Paul Bernays*, 14th ed. by Michael Toepell, Stuttgart/Leipzig: B. G. Teubner.

KERRY, Benno *1890* "Ueber Anschauung und ihre psychische Verarbeitung (Siebenter Artikel)," *Vierteljahrsschrift für wissenschaftliche Philosophie* **14**, 317–343.

MADDY, Penelope *1990 Realism in Mathematics*, Oxford: Clarendon Press.

— *1997 Naturalism in Mathematics*, Oxford: Clarendon Press.

MANCOSU, Paolo *1998 From Brouwer to Hilbert. The Debate on the Foundations of Mathematicsin the 1920s*, New York/Oxford: Oxford University Press.

MEHRTENS, Herbert *1990 Moderne, Sprache, Mathematik. Eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme*, Frankfurt a. M.: Suhrkamp.

NELSON, Leonard *1904* "Die kritische Methode und das Verhältnis der Psychologie zur Philosophie. Ein Kapitel aus der Methodenlehre," *Abhandlungen der Fries'schen Schule* n. s. **1**, issue 1, 1–88.

— *1906* "Kant und die Nicht-Euklidische Geometrie," *Das Weltall* **6**, 147–155, 174–182, 186–193.

— *1928* "Kritische Philosophie und mathematische Axiomatik," *Unterrichtsblätter für Mathematik und Naturwissenschaften* **34**, 108–142.

PAPPUS VON ALEXANDRIA *1986 Book 7 of the* Collection, ed. Alexander Jones, vol. 1, Springer: New York etc. (*Sources in the History of Mathematics and Physical Sciences*; 8).

PECKHAUS, Volker *1990 Hilbertprogramm und Kritische Philosophie. Das Göttinger Modell interdisziplinärer Zusammenarbeit zwischen Mathematik und Philosophie*, Göttingen: Vandenhoeck & Ruprecht 1990 (*Studien zur Wissenschafts-, Sozial- und Bildungsgeschichte der Mathematik*; 7)

— *1996* "The Axiomatic Method and Ernst Schröder's Algebraic Approach to Logic," *Philosophia Scientiae. Traveaux d'histoire et de philosophie des science* (Nancy) 1/3 (1996), 1–15.

— *1997 Logik, Mathesis universalis und allgemeine Wissenschaft. Leibniz und die Wiederentdeckung der formalen Logik im 19. Jahrhundert*, Berlin: Akademie-Verlag 1997 (*Logica Nova*).

QUINE, W. V. *1975* "Five milestones of empiricism," repr. in Quine, *Theories and Things*, Cambridge, Mass: Harvard University Press 1981, 67–72.

REID, Constance *1970 Hilbert. With an Appreciation of Hilbert's Mathematical Work by Hermann Weyl*, New York/Heidelberg/Berlin: Springer.

SCHRÖDER, Ernst *1873 Lehrbuch der Arithmetik und Algebra für Lehrer und Studirende*, vol. 1 [no further volumes]: *Die sieben algebraischen Operationen*, Leipzig: B. G. Teubner.

SCHUBERT, Hermann *1898* "Grundlagen der Arithmetik," in *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, vol. 1:

Arithmetik und Algebra, ed. Wilhelm Franz Meyer, pt. 1, Leipzig: Teubner 1898–1904, 1–27 [published 7 November 1898].

Sieg, Wilfried *1999* "Hilbert's Programs: 1917–1922," *Bulletin of Symbolic Logic* **5**, 1–44.

Wiechert, Emil *1899* "Grundlagen der Elektrodynamik," in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*, ed. by the Fest-Comitee, Leipzig.