Paradoxes in Göttingen

Volker Peckhaus*

1 Introduction

In 1903 Russell's paradox came over the mathematical world with a double stroke. Bertrand Russell himself published it under the heading "The Contradiction" in chapter 10 of his *Principles of Mathematics* (Russell *1903*). Almost at the same time Gottlob Frege (1848–1925) referred to Russell's paradox in the postscript of the second and final volume of his *Grundgesetze der Arithmetik* (Frege *1903*), admitting that the foundational system of the *Grundgesetze* had proved to be inconsistent.

Frege sent a copy of this volume to his Göttingen colleague David Hilbert (1862–1943), and received an astonishing reply. Hilbert assured that "this example," as he called the paradox, had already been known before in Göttingen. In a footnote he added, "I believe Dr Zermelo discovered it three or four years ago after I had communicated my examples to him," and continued

> I found other even more convincing contradictions as long as four or five years ago; they led me to the conviction that traditional logic is inadequate and that the theory of concept formation needs to be sharpened and refined.[1]

If this was true, much of the fame of Russell's paradox would have gone. So several questions can be raised:

- What exactly were these paradoxes said to have been known in Göttingen already before the publication of Russell's paradox?
- What was Ernst Zermelo's (1871–1953) role in this story?

---

*Universität Paderborn, Fakultät für Kulturwissenschaften – Philosophie, Warburger Str. 100, D – 33098 Paderborn, Germany, e-mail: peckhaus@hrz.upb.de. I would like to thank Marcello Ghin, Paderborn, for helpful comments on a previous version. This paper very much profited from an earlier collaboration with Reinhard Kahle on "Hilbert's Paradox", the result of which has been published as Peckhaus/Kahle *2002*.

[1]Frege *1980*, 51. German original in Frege *1976*, 79–80.

- Why did Hilbert and Zermelo not publish the paradoxes they found?
- What was the impact of Russell's paradox, if any, in Göttingen?
- In particular, who discussed the paradox in Göttingen? Did this discussion transcend the borders of the mathematical discourse?

Most of these questions have already been answered convincingly. These answers show that the standard story of the history of set theory according to which Russell's paradox led to the axiomatization of set theory is as false as the assumption that at least in Göttingen there was no impact at all, an assumption which could be drawn from Hilbert's self-assured statement in his letter to Frege. The impact was, however, not like a stroke that immediately changed the mathematical world in pre and post paradox mathematics. In any case, the publication led to new insights into the nature of mathematical systems, sometimes not at once but after a while of thinking it over.

This paper will not present new historical material on the impact of Russell's paradox in Göttingen. It aims at surveying and combining the results of previous research on the matter, thereby illustrating the complexity of the story of the paradoxes.

The paper will be divided into two parts. In the first part the period up to 1903 will be considered, especially Hilbert's exchange with Cantor on unintended sets, and the paradoxes of Hilbert and Zermelo. The second part is devoted to the effects caused by the publication of Russell's paradox. It will cover the period between 1903 and 1908. In this part the discussion of Russell's paradox among Göttingen philosophers standing close to the Hilbert circle will also be presented.

2 The Paradoxes up to 1903

2.1 Cantor's Paradox

Hilbert's first contact with the so-called "the paradoxes of set theory" can be traced back to his correspondence with Georg Cantor (1845–1918) between 1897 and 1900.[2] Already in the first of his letters, dated 26 September 1897 (Cantor *1991*, no. 156, 388–389), Cantor proved that the totality of alephs does not exist, i. e., that this totality is not a well-defined, finished set [*fertige Menge*]. If it is taken to be a finished set, a certain larger aleph would follow on this totality. So this new aleph would at the same time belong to the totality of all alephs, and not belong to it, because of being larger than all alephs (*ibid.*, 388).

[2]For a comprehensive discussion of this correspondence cf. Purkert/Ilgauds *1987*, 147–166. Extracts are published in Cantor *1991*.

Hilbert and Cantor were thus concerned with what later became known under the name "Cantor's paradox". But in fact, Cantor was not really dealing with paradoxes and their solution, but with non-existence proofs using *reductio-ad-absurdum* arguments.[3] He disproved the existence of the totality of all cardinals by showing that the assumption of its existence contradicts his definition of a set as the comprehension of certain well distinguished objects of our intuition or our thinking in a whole (Cantor *1895*, cf. Cantor *1932*, 282). The totality of all cardinals (and of all ordinals) cannot be thought of as *one* such thing, contrary to actual infinite objects like transfinite sets.

Hilbert's letters to Cantor have not been preserved, but we can derive Hilbert's opinion from published talks. These talks show that Hilbert took the doubts about the existence of the set of all cardinals seriously, obviously not convinced by Cantor's non-existence proof. Hilbert saw an alternative to this proof in a suitable axiomatization of set theory. In the paper "On the Concept of Number" from 1900 (Hilbert *1900b*), Hilbert's first paper on the foundations of arithmetic, he gave a set of axioms for arithmetic, and claimed that only a suitable modification of known methods of inference would be needed for proving the consistency of the axioms. If this proof were successful, the existence of the totality of real numbers would be shown at the same time. In this context he referred to Cantor's problem of whether the system of real numbers is a consistent, or finished, set. He stressed:

> Under the conception above, the doubts which have been raised against the existence of the totality of all real numbers (and against the existence of infinite sets in general) lose all justification; for by the set of real numbers we do not have to imagine the totality of all possible laws according to which the elements of a fundamental sequence can proceed, but rather—as just described—a system of things whose internal relations are given by a *finite and closed* set of axioms [...], and about which new statements are valid only if one can derive them from the axioms by means of a finite number of logical inferences.[4]

He also claimed that the existence of the totality of all powers or of all Cantorian alephs could be disproved, i. e., in Cantor's terminology, that the system of all powers is an inconsistent (not finished) set (*ibid.*).

Hilbert took up this topic again in his famous 1900 Paris lecture on "Mathematical Problems".[5] In the context of his commentary on the second problem concerning the consistency of the axioms for arithmetic he used the same examples from Cantorian set theory and the continuum problem as

[3]We follow in this evaluation G. H. Moore and A. Garciadiego (Moore/Garciadiego *1981*, Garciadiego Dantan *1992*).

[4]Hilbert *1996b*, 1095. German original Hilbert *1900b*, 184.

[5]Hilbert *1900a*, English translations Hilbert *1902*, Hilbert *1996a*.

in the earlier lecture. “If contradictory attributes be assigned to a concept,” he wrote, “I say, that mathematically the concept does not exist” (Hilbert *1996a*, 1105).

According to Hilbert a suitable axiomatization would avoid the contradictions resulting from the attempt to comprehend absolute infinite multiplicities as units, because only those concepts had to be accepted which could be deduced from an axiomatic basis.

2.2 Hilbert's Paradox

Hilbert's remarks at these prominent places show that he was worried by the problems causing Cantor to distinguish between finished or consistent sets and multiplicities that are no sets seriously. Unrestricted comprehension, a natural way of forming sets, had been removed by a stipulation made necessary by the fact that a contradiction would arise if that is not done. A suitable axiomatization, however, would exclude these problematic multiplicities from the outset. Hilbert was worried not simply by the fact that these contradictions occurred in set theory, but by the pragmatic aspect that methods used by mathematicians as a matter of course had proved to lead to unintended, contradictory results. And this pragmatic aspect furthermore shows that not only Cantorian set theory could be concerned, but also other domains of mathematics. Hilbert gave evidence to this by formulating a paradox of his own which was “purely mathematical” in the sense that it didn't use notions from transfinite set theory. This paradox was known in Göttingen as “Hilbert's paradox”, the one he obviously referred to in his letter to Frege, and which Otto Blumenthal mentioned in his biographical note in the third volume of Hilbert's *Collected Works* (Blumenthal *1935*).[6]

Hilbert never published the paradox. He discussed it, however, in a lecture course on the “Logical Principles of Mathematical Thinking” he gave in Göttingen in the summer term of 1905. It is preserved in two sets of notes (Hilbert *1905b*, Hilbert *1905c*). Part B of these notes, on “The Logical Foundations”, starts with a comprehensive discussion of the paradoxes of set theory. It begins with metaphorical considerations on the general development of science:

> It was, indeed, usual practice in the historical development of science that we began cultivating a discipline without many scruples, pressing onwards as far as possible, that we thereby, however, then ran into difficulties (often only after a long time) that forced us to turn back and reflect on the foundations of the discipline. The house of knowledge is not erected like a dwelling where the foundation is first well laid-out

[6] On the history of Hilbert's paradox cf. Peckhaus *1990b*, Peckhaus/Kahle *2002*.

> before the erection of the living quarters begins. Science prefers to obtain comfortable rooms as quickly as possible in which it can rule, and only subsequently, when it becomes clear that, here and there, the loosely joined foundations are unable to support the completion of the rooms, science proceeds in propping up and securing them. This is no shortcoming but rather a correct and healthy development.[7]

Although contradictions are quite common in science, Hilbert continued, in the case of set theory they seem to be different, because there they have a tendency towards the side of theoretical philosophy. In set theory the common Aristotelian logic and its standard methods of concept formation had been used without hesitation. And these standard tools of purely logical operations, especially the subsumption of concepts under a general concept, now proved to be responsible for the new contradictions.

Hilbert elucidated these considerations by presenting three examples, the Liar paradox, "Zermelo's paradox," as the Russell-Zermelo paradox was called in Göttingen at that time which will be discussed below, and the paradox of his own, which was, according to Hilbert, of purely mathematical nature (Hilbert *1905b*, 210). Hilbert expressed his opinion that this paradox

> appears to be especially important; when I found it, I thought in the beginning that it causes invincible problems for set theory that would finally lead to the latter's eventual failure; now I firmly believe, however, that everything essential can be kept after a revision of the foundations, as always in science up to now. I have not published this contradiction, but it is known to set theorists, especially to G. Cantor.[8]

The paradox is based on a special notion of set which Hilbert introduces by means of two set formation principles starting from the natural numbers. The first principle is the *addition principle*. In analogy to the finite case, Hilbert argued that the principle can be used for uniting two sets together "into a new conceptual unit [. . .], a new set that contains each element of either sets." This operation can be extended: "In the same way, we are able to unite several sets and even infinitely many into a union." The second principle is called the *mapping principle*. Given a set $\mathcal{M}$, he introduces the set $\mathcal{M}^{\mathcal{M}}$ of *self-mappings* of $\mathcal{M}$ to itself.[9] A self-mapping is just a total function which maps the elements of $\mathcal{M}$ to elements of $\mathcal{M}$.[10]

Now, he considers all sets which result from the natural numbers "by applying the operations of addition and self-mapping an arbitrary number of

[7] Hilbert *1905c*, 122, published in Peckhaus *1990b*, 51.

[8] Hilbert *1905b*, 204, published in Peckhaus *1990b*, 52.

[9] Hilbert used the German term "*Selbstbelegung*" which is translated here by "self-mapping".

[10] In classical logic, $\mathcal{M}^{\mathcal{M}}$ is isomorphic to $2^{\mathcal{M}}$, and the set of all mappings from $\mathcal{M}$ to $\{0, 1\}$ is isomorphic to $\mathcal{P}(\mathcal{M})$, the power set of $\mathcal{M}$.

times." By use of the addition principle which allows to build the union of arbitrary sets one can "unite them all into a sum set $\mathcal{U}$ which is well-defined." In the next step the mapping principle is applied to $\mathcal{U}$, and we get $\mathcal{F} = \mathcal{U}^{\mathcal{U}}$ as the set of all self-mappings of $\mathcal{U}$. Since $\mathcal{F}$ was built from the natural numbers by using the two principles only, Hilbert concludes that it has to be contained in $\mathcal{U}$. From this fact he derives a contradiction.

Since "there are 'not more' elements" in $\mathcal{F}$ than in $\mathcal{U}$ there is an assignment of the elements u_i of $\mathcal{U}$ to elements f_i of $\mathcal{F}$ such that all elements of f_i are used. Now one can define a self-mapping g of $\mathcal{U}$ which differs from all f_i. Thus, g is not contained in $\mathcal{F}$. Since $\mathcal{F}$ was assumed to contain all self-mappings we have a contradiction. In order to define g Hilbert used Cantor's diagonalization method. If f_i is a mapping u_i to $f_i(u_i) = u_{f_i^{(i)}}$ he chooses an element $u_{g^{(i)}}$ different from $u_{f_i^{(i)}}$ as the image of u_i under g. Thus, we have $g(u_i) = u_{g^{(i)}} \neq u_{f_i^{(i)}}$ and g "is distinct from any mapping f_k of $\mathcal{F}$ in at least one assignment."[11]

Hilbert finishes his argument with the following observation:

> We could also formulate this contradiction so that, according to the last consideration, the set $\mathcal{U}^{\mathcal{U}}$ is always bigger [of greater cardinality][12] than $\mathcal{U}$ but, according to the former, is an element of $\mathcal{U}$.

Hilbert's paradox is closely related to Cantor's paradox. Both, Cantor and Hilbert, construct "sets" which lead to contradictions. This is shown with the help of Cantor's diagonalization argument. However, the ways in which these "sets" are constructed differ essentially. According to Cantor (Cantor *1883*, § 11, cf. Cantor *1932*, 195–197), there are three principles for the generation of cardinals. The first principle ("erstes Erzeugungsprinzip") concerns the generation of real whole numbers [*reale ganze Zahlen*, i. e., ordinal numbers] by adding a unit to a given, already generated number. The second principle allows the formation of a new number, if a certain succession of whole numbers with no greatest number is given. This new number is imagined as the limit of this succession. Cantor adds a third principle, the inhibition or restriction principle ("Hemmungs- oder Beschränkungsprinzip") which grants that the second number class has not only a higher cardinality than the first number class, but exactly the next higher cardinality. Considering Cantor's general definition of a set as the comprehension of certain well-distinguished objects of our intuition or our thinking as a whole (Cantor *1895*, Cantor *1932*, 282), one can justly ask whether the sets of all cardinals, of all ordinals or the universal set of all sets are sets according to this definition, i. e.,

[11] Hilbert's notation $u_{g^{(i)}}$ is somewhat clumsy. In fact, it is enough to say that $g(u_i) = v_i$ for an element v_i of $\mathcal{U}$ with $v_i \neq f_i(u_i)$.

[12] Remark later added in Hilbert's hand in Hellinger's lecture notes.

whether an unrestricted comprehension is possible. Cantor denies this, justifying his opinion with the help of a *reductio ad absurdum* argument, but he doesn't exclude the possibility of forming the paradoxes by provisions in his formalism.

Hilbert, on the other hand, introduces two alternative set formation principles, the addition principle and the mapping principle, but they lead to paradoxes as well. In avoiding concepts from transfinite arithmetic Hilbert believes that the purely mathematical nature of his paradox is guaranteed. For him, this paradox appears to be much more serious for mathematics than Cantor's, because it concerns an operation that is part of everyday practice of working mathematicians.

2.3 Zermelo's Paradox

Ernst Zermelo came to Göttingen in 1897 in order to work for his *Habilitation*. His special fields of competence were the calculus of variations and mathematical physics, such as thermodynamics and hydrodynamics.[13] Under the influence of Hilbert he changed the focus of his interests to set theory and foundations. He became Hilbert's collaborator in the foundations of mathematics. His first set-theoretical publication on the addition of transfinite cardinals dates from 1901 (Zermelo *1901*), but as early as the winter term 1900/1901 he gave a lecture course on set theory in Göttingen. It is possible that he may have found the paradox while preparing this course. He referred to it in the famous polemical paper "A New Proof of the Possibility of a Well-Ordering" of 1908 (Zermelo *1908a*). There Zermelo noted that he had found the paradox independently of Russell, and that he had mentioned it to Hilbert and other people already before 1903. And indeed, among the papers of Edmund Husserl (1859–1938), until 1916 professor of philosophy in Göttingen, a note in Husserl's hand was found, partially written in Gabelsberger shorthand, saying that Zermelo had informed him on 16 April 1902 that the assumption of a set M that contains all of its subsets m, m', ... as elements, is an inconsistent set, i. e., a set which, if treated as a set at all, leads to contradictions.[14] Zermelo's message was a comment on a review that Husserl had written on the first volume of Ernst Schröder's (1841–1902) *Vorlesungen über die Algebra der Logik* (Schröder *1890*). Schröder had criticized George Boole's interpretation of the symbol 1 as the class of everything that

[13] On Zermelo's activities in Göttingen cf. esp. Moore *1982*, Peckhaus *1990a*, Peckhaus *1990b*, 76–122.

[14] Critical edition in *Husserliana* XXII (Husserl *1979*, 399). English translation in Rang/Thomas *1981*.

can be a subject of discourse (the universe of discourse, universal class).[15] Husserl had dismissed Schröder's argumentation as sophistical (Husserl *1891*, 272), and was now advised by Zermelo that Schröder was right concerning the matter, but not in his proof.

In own recollections communicated to Heinrich Scholz in 1936, Zermelo saw the origins of his paradox in discussions in the Hilbert circle. At that time Heinrich Scholz was working on the papers of Gottlob Frege which he had acquired for his department at the University of Münster. He had found Hilbert's letter to Frege, mentioned above, and now asked Zermelo what paradoxes Hilbert referred to in this letter.[16] Zermelo answered that the set-theoretic paradoxes were often discussed in the Hilbert circle around 1900, and he himself had given at that time a precise formulation of the paradox which was later named after Russell.[17]

As mentioned above, Hilbert discussed it in his 1905 lectures course presenting it as a "purely logical" example probably more convincing for non-mathematicians whereas his own (purely mathematical) example seamed to be more decisive for mathematicians, as Hilbert claimed (Hilbert *1905b*, 210).

3 Thinking the Paradoxes Over, 1903–1908

3.1 The Mathematicians' Reaction on Russell's Paradox

Given the naive attitude towards the contradictions of set theory, the publication of Russell's paradox could not easily be ignored because the paradox unveiled the consequences of this special kind of contradictions for logic. Before 1903 Hilbert suggested to avoid or circumvent the contradictions with the help of an axiomatic reformulation of set theory. This suggestion was kept after 1903, but he had to extend it considerably. His axiomatic programme was deeply concerned since its germ, the consistency proof for the arithmetical axioms by means of mathematical and logical standard methods, required logic itself to be free of contradictions.

In his seminal "Grundlagen der Geometrie" (1899) Hilbert had proved the consistency of Euclidean geometry under the provision of the consistency of arithmetic (cf. Hilbert *1899*). The consistency of arithmetic, however, had still to be shown. In his talk "Über den Zahlbegriff" (1900) Hilbert claimed that the consistency proof for arithmetic only required a suitable modifica-

[15] Schröder *1890*, 245. Schröder referred to Boole's definition of the universe of discourse and his interpretation of the symbol 1, cf. Boole *1854*, 42–43.

[16] Heinrich Scholz to Zermelo, dated Münster, 5 April 1936, University Archive Freiburg i. Br., Zermelo papers, C 129/106.

[17] Zermelo to Scholz, dated Freiburg i. Br., 10 April 1936, Institut für mathematische Logik und Grundlagenforschung, Münster, Scholz papers.

tion of known methods of inference (Hilbert *1900b*, 184), an opinion which soon proved to be overoptimistic. In the Paris problems talk, he included the consistency proof for arithmetic as the second among the problems discussed (Hilbert *1900a*). Frege's disappointed admission made now obvious that this consistency proof could not be done using means just proved to be inconsistent.

The talks on foundations of mathematics and logic at meetings of the Göttingen Mathematical Society give evidence for the new activities released by the publication of the paradoxes.[18] In 1901 and 1902 the main speaker was Hilbert himself giving papers on special problems in the axiomatic foundation of geometry. Furthermore, Edmund Husserl gave his "double lecture" on completeness and definiteness of axiomatic systems (26 November and 10 December 1901). In 1903 Ernst Zermelo discussed Frege's concept of number as presented in the second volume of the *Grundgesetze der Arithmetik* (12 May 1903). Obviously the Göttingen mathematicians felt a need to go deeper into Frege's failed theory. Hilbert spoke about the axiomatic standpoint in the foundations of arithmetic emphasizing the principle of contradiction as the "pièce de résistance" (27 December 1903). Early in 1904 H. Fleischer made the Göttingen mathematicians familiar with Italian positions in foundations, reporting, e. g., on Giuseppe Peano's (1858–1932) *Arithmetices principia, nova methododo exposita* (Peano *1889*) (19 January and 23 February 1904). Several lectures on the axiomatization of set theory were given, especially by E. Zermelo in 1904, 1906 and 1908. And on 7 February 1905 William Henry Young (1863–1942) reported on Russell's *Principles of Mathematics* and Russell's paradox.

These activities are indications for a deep revision of Hilbert's axiomatic programme. The programme before 1903 consisted in axiomatizing certain fields of mathematics in which foundations had been questioned, reducing their consistency to the consistency of arithmetic. Set theory was not an integral methodological part of this programme, but was among the fields of mathematics waiting for being axiomatized. After 1903 it became clear that the axiomatization of arithmetic required an axiomatization of logic and set theory. One precondition was to gain competence in modern logic, i. e., both, the Fregean mathematical logic and G. Boole's and E. Schröder's algebra of logic. Therefore logic, formerly regarded as a basic philosophical subdiscipline, came into the focus of the Göttingen mathematicians. At the instigation of Hilbert Zermelo was awarded with a stipendiary lectureship on "Mathematical Logic and Related Fields" in 1907. Zermelo delivered the first German lecture course on mathematical logic which was based on a

[18]According to the reports in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* **11** (1901)–**14** (1905).

ministerial commission, in the summer term of 1908 (cf. Peckhaus *1990a*, *1990b*, *1992*, *1994a*).

Now logic and set theory were integrated into the axiomatic programme, i. e., securing the logical and set-theoretical grounds was seen as a necessary precondition for proving the consistency of arithmetic. Arithmetic was treated in a rather logicistic manner, i. e., as soon as the consistency of logic and set theory (the dealing with the infinite in mathematics) was shown, there would only be one step more to prove the consistency of arithmetic. This was expressed by Hilbert in his rather obscure demand of "a partially simultaneous development of the laws of logic and arithmetic" called for in his Heidelberg talk on the foundations of logic and arithmetic, Hilbert's first published reaction to the paradoxes (Hilbert *1905a*, 170). These ideas were considerably deepened in the lecture course on the logical principles of mathematical thinking mentioned above.

Hilbert's original approach to reduce the consistency of any set of mathematical axioms to the consistency of arithmetic was thus replaced by a three step programme to create consistent sets of axioms for logic, set theory and then arithmetic. One of the first fruits of this revised programme was Ernst Zermelo's formulation of the set-theoretic axioms published in 1908, although Zermelo had to confess that he was not yet able to prove "the 'consistency,' no doubt very essential, of my axioms" (Zermelo *1908b*, 262). Zermelo, thus, only executed an old demand of Hilbert's from before 1903.

3.2 The Philosophical Discussion in the Nelson Circle

One of the participants in Hilbert's summer lecture course of 1905 was the 23 year-old philosopher Leonard Nelson (1882–1927), who had received his doctorate in philosophy in Göttingen in July 1904.[19] He was already head of a philosophical school, the *Neue Fries'sche Schule* that was devoted to critical philosophy in the spirit of Jacob Friedrich Fries (1773–1843). Nelson was still a student when he founded the new series of the *Abhandlungen der Fries'schen Schule* as a forum for his circle, assisted by his older friend Gerhard Hessenberg (1873–1925), then lecturer for mathematics at the Academy for Military Technology in Charlottenburg near Berlin, and by the physiologist Karl Kaiser (1861–c. 1933). Hessenberg was a known geometer, but he also gain recognition as a set theorist after having published the first textbook on set theory (Hessenberg *1906*).

In June 1905 Nelson sent a letter to Hessenberg commenting on Hilbert's lecture "Über die Grundlagen der Arithmetik" (Hilbert *1905a*), and he ex-

[19] For more details on Nelson and his circle cf. Peckhaus *1990b*, 123–154, with further hints on literature. On the philosophical discussion of the paradoxes in Göttingen, especially the emergence of Grelling's paradox see Peckhaus *1995*.

pressed his disappointment about Hilbert's ideas. Rather perplexed he wrote: "To remove the contradictions in set theory, he [i.e., Hilbert] intends to reform (not set theory but) logic. Well, we shall see, how he will do it."[20] Hessenberg answered quite to the point:[21]

> I do not at all consider it as paradoxical that one has to reform logic in order to make set theory free of contradictions. First of all it is not yet possible to separate logic sharply from arithmetical considerations. Secondly, however: If there are paradoxes in set theory, then either the inferences are not correct or the concepts generated are contradictory.

In both cases, Hessenberg continued, it is a logical task to uncover the mistakes. According to the laws of logic a thing a falls under the concept b or not. No other principle is needed for the concept of a set. Hessenberg stressed that Hilbert very much strengthened the requirements for building concepts in order to avoid the resulting paradoxes.

In Hilbert's lecture course Nelson learned more about the paradoxes, not about Russell's, but about its Göttingen variation, Zermelo's paradox, and about Hilbert's paradox. It is instructive to observe the growing significance of Russell's name in the correspondence between Hessenberg and Nelson. In June 1905 Hessenberg, who was at that time going to write his textbook on set theory, recalled that Nelson had spoken about "Hilbert's paradox of the set of sets that belong to themselves," obviously misconceiving what Nelson had told him.[22] In the same month Nelson directed Hessenberg's interest towards Russell's *Principles of Mathematics*, a book in which the foundations of all mathematical disciplines and especially of set theory were discussed, as Nelson wrote.[23] In February 1906 Nelson sent Russell's book to Hessenberg whose judgement on its set-theoretical parts was, however, scathing. He had found almost nothing that made any impression on him. He regarded Russell's logicistic standpoint (contrary to that of Dedekind) as utterly ridiculous, and criticized that Russell expatiated "the completely vague contradiction of the set of all sets not being subsets of themselves in a whole chapter, but found only a few unimportant words on the considerable contradiction of the set of all ordinals."[24]

[20] Nelson to Hessenberg, dated Göttingen, 16 June 1905, Bundesarchiv, Abt. Potsdam, Nelson Papers, 90 Ne 1, fol. 50–53.

[21] Hessenberg to Nelson, dated Grunewald, 26 June 1905, Archiv der sozialen Demokratie, Nelson Papers, 1/LN AA 000270.

[22] Hessenberg to Nelson, dated Grunewald, 26 June 1905, Archiv der sozialen Demokratie, Nelson Papers, *ibid.*

[23] Nelson to Hessenberg, dated Grunewald, 7 February 1906, Archiv der sozialen Demokratie, Nelson Papers, 1/LN AA 000271.

[24] Hessenberg to Nelson, dated Grunewald, 7 February 1906, Bundesarchiv, Abt. Potsdam, 90 Ne 1, no. 389, fol. 54*f*.

It is astonishing to see that, although the paradoxes became a widely discussed topic, no one in the beginning made efforts to study the relevant books of Russell or Frege in greater detail, in Nelson's circle at least not before late 1905 or early 1906.

In contrast to Hessenberg, the paradoxes did electrify the more philosophically minded members of Nelson's circle. In June 1906 Heinrich Goesch (1880–1930) (cf. Peckhaus *1990b*, 137–140) wrote the following letter to his friend Leonard Nelson (mentioning other members of Nelson's circle):[25]

> Grelling informed me of Russel[l]'s paradox in logic concerning the concept impredicable. I succeeded in solving the same, and then I learned from Berkowski of another paradox also from Russel[l] in set theory, concerning the concept of a set not belonging to itself. For this paradox my solution holds as well. Therefore Berkowski, who told me that the mathematicians working in set theory have no solution for the paradox up to now, thought that the matter might be not unimportant. I therefore would like to write a short paper, and I would like to ask you to tell me in which book of Russel[l]'s these paradoxes can be found and to which German presentations [Ausformungen] one should refer. I think that the matter will be finished in a few days.

Of course, Goesch's paper was not written in a few days. A first version of the announced manuscript was not completed before spring 1907.[26] Nelson was sceptical and commissioned his closest collaborator, the mathematics student Kurt Grelling (1886–1943)[27] to check Goesch's ideas.

In this period of discussion several other members of the circle were involved in attempts to solve the paradoxes. One of them was Otto Meyerhof (1884–1951),[28] the 1923 Nobel laureate for medicine and physiology; another was Alexander Rüstow (1885–1963),[29] after World War II one of the fathers of the German social economy. Rüstow asked Nelson whether he could publish his doctoral thesis in the *Abhandlungen der Fries'schen Schule*.[30] It had the characteristic title *Der Lügner. Theorie, Geschichte und Auflösung des Russellschen Paradoxons* (*The Liar. Theory, History, and Solution of Russell's Paradox*). Nelson refused, because he rejected Rüstow's solution.[31] Rüstow's

[25] Goesch to Nelson, undated (Munich, 14 June 1906), Archiv der sozialen Demokratie, Nelson Papers, 1/LN AA 000255.

[26] In the correspondence with Nelson (Archiv der sozialen Demokratie, Nelson Papers, box 27) a postal receipt of delivery for a manuscript can be found, dated 23 April 1907.

[27] On Grelling's tragical biography cf. Peckhaus *1993*, *1994b*.

[28] Cf. Peckhaus *1990b*, 135–137.

[29] Cf. Peckhaus *1990b*, 140–142.

[30] Cf. Nelson to Hessenberg, dated Westend, 23 January 1908, Bundesarchiv, Abt. Potsdam, 90 Ne 1, no. 389, fol. 155–157.

[31] Nelson to Rüstow, dated Göttingen, 9 February 1908, Bundesarchiv Koblenz, NL 169, Rüstow.

thesis was published not before 1910, and it is also characteristic that in the title of the published version the reference to Russell's paradox was omitted (Rüstow *1910*).

It was during Grelling's and Nelson's struggle with Goesch's "solution" that Grelling and Nelson compiled material for a joint paper which was finally published in 1908 (Grelling/Nelson *1908*). They looked for the basic logical conditions for the occurrences of the paradoxes, and distinguished between

> the task of a proper "solution" of the paradox, i.e., the task of unveiling the underlying appearance, and the task of a "correction," i.e., the task of avoiding the paradox by introducing new, consistent concepts. Such a correction can not be considered to be a solution, because the paradoxical objects, if they exist at all, are not eliminated by stopping work on them.[32]

Most importantly Grelling discovered new paradoxes, among them the semantical heterological paradox, today known under Grelling's name. It runs in its original version as follows (Grelling/Nelson *1908*, 307):

> Let $\varphi(M)$ be the word that denotes the concept defining M. This word is either an element of M or not. In the first case we will call it "autological" in the other "heterological."[33] Now the word "*heterological*" is itself either autological or heterological. Suppose it to be autological; then it is an element of the set defined by the concept that is denoted by itself, hence it is heterological, contrary to the supposition. Suppose, however, that it is heterological; then it is not element of the set defined by the concept that is denoted by itself, hence it is not heterological, again against the supposition.

It is this Grelling's paradox which Frank Plumpton Ramsey (1903–1930, in Ramsey *1926*) wrongly attributed to Hermann Weyl (1885–1955), who had mentioned it in *Das Kontinuum* as "a well-known paradox, essentially coming from Russell" and had discussed it as "scholasticism of the worst kind" (Weyl *1918*, 2). History has shown that Weyl's judgement did not do justice to the importance of Grelling's paradox.

References

Blumenthal, Otto *1935* "Lebensgeschichte," in David Hilbert, *Gesammelte Abhandlungen*, vol. III, Springer, Berlin, 388–429. Second edition, Springer, Berlin/Heidelberg/New York, 1970.

[32] Grelling/Nelson *1908*, 314.

[33] "Short", e.g., is autological, "long" heterological; "English" is autological, "German" heterological.

BOOLE, George *1854 An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*, Walton & Maberly, London. Reprinted Dover, New York, c. 1958.

CANTOR, Georg *1883* "Ueber unendliche, lineare Punktmannichfaltigkeiten [5th pt.]," *Mathematische Annalen* **21**, 545–591. Reprinted in Cantor *1932*, 165–209.

— *1895* "Beiträge zur Begründung der transfiniten Mengenlehre," *Mathematische Annalen* **46/49** (1895/97), 207–246. Reprinted in Cantor *1932*, 282–351.

— *1932 Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, ed. Ernst Zermelo, Springer, Berlin. Reprinted Olms, Hildesheim 1962; Springer, Berlin/Heidelberg/New York, 1980.

— *1991 Briefe*, eds. H. Meschkowski/W. Nilson, Springer, Berlin et al.

FREGE, Gottlob *1903 Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, vol. 2, Hermann Pohle, Jena. Reprinted together with vol. 1, Olms, Hildesheim, 1966.

— *1976 Wissenschaftlicher Briefwechsel*, eds. G. Gabriel et al., Felix Meiner, Hamburg.

— *1980 Philosophical and Mathematical Correspondence*, eds. G. Gabriel et al., Basil Blackwell, Oxford.

GARCIADIEGO DANTAN, Alejandro R. *1992 Bertrand Russell and the Origins of the Set-theoretic "Paradoxes"*, Birkhäuser, Basel/Boston/Berlin.

GRELLING, Kurt/NELSON, Leonard *1908* "Bemerkungen zu den Paradoxieen von Russell und Burali-Forti," *Abhandlungen der Fries'schen Schule*, n. s. **2**, no. 3, 301–334.

HESSENBERG, Gerhard *1906* "Grundbegriffe der Mengenlehre. Zweiter Bericht über das Unendliche in der Mathematik," *Abhandlungen der Fries'schen Schule* n. s.,**1**, no. 4, 479–706.

HILBERT, David *1899* "Grundlagen der Geometrie," in *Festschrift zur Feier der Enthüllung des Gauss-Weber-Denkmals in Göttingen*, B. G. Teubner, Leipzig, 1–92. Recent edition Hilbert *1999*.

— *1900a* "Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900," *Nachrichten von der königl. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse aus dem Jahre 1900*, 253–297. English translation Hilbert *1902*, extract Hilbert *1996a*.

— *1900b* "Über den Zahlbegriff," *Jahresbericht der Deutschen Mathematiker-Vereinigung* **8**,180–184. English translation Hilbert *1996b*.

— *1902* "Mathematical Problems. Lecture Delivered Before the International Congress of Mathematicians at Paris in 1900", translated by Mary Winston Newson, *Bulletin of the American Mathematical Society* **8**, 437–479. Translation of Hilbert *1900a*.

— *1905a* "Über die Grundlagen der Logik und der Arithmetik," in Adolf Krazer (ed.), *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13. August 1904*, B. G. Teubner, Leipzig, 174–185.

— *1905b Logische Principien des mathematischen Denkens*, lecture course in the summer term 1905, lecture notes by Ernst Hellinger, Library of the Mathematics Seminar of the University of Göttingen.

— *1905c Logische Principien des mathematischen Denkens*, lecture course in the summer term 1905, lecture notes by Max Born, Niedersächsische Staats- und Universitätsbibliothek Göttingen, Cod. Ms. D. Hilbert 558a.

— *1996a* "*From* Mathematical Problems," in W. Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, vol. 2, Clarendon Press, Oxford, 1096–1105.

— *1996b* "On the Concept of Number," in W. Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics*, vol. 2, Clarendon Press, Oxford, 1089–1095.

— *1999 Grundlagen der Geometrie. Mit Supplementen von Paul Bernays*, 14th ed. by M. Toepell, B. G. Teubner, Stuttgart/Leipzig.

Husserl, Edmund *1891* Review of Schröder *1890*, *Göttingische gelehrte Anzeigen*, 243–278; critical edition in Husserl *1979*, 3–43.

— *1979 Aufsätze und Rezensionen (1890–1910), mit ergänzenden Texten*, Husserliana, Vol. XXII, ed. B. Rang, Nijhoff, The Hague/Boston/London.

Moore, Gregory H./Garciadiego, Alejandro *1981* "Burali-Forti's Paradox: A Reappraisal of Its Origins," *Historia Mathematica* **8**, 319–350.

Moore, Gregory H. *1982 Zermelo's Axiom of Choice. Its Origins, Development and Influence*, Springer, New York/Heidelberg/Berlin(*Studies in the History of Mathematics and Physical Sciences*; 8.

Peano, Giuseppe *1889 Arithmetices principia, nova methododo exposita*, Bocca, Augustae Taurinorum.

Peckhaus, Volker *1990a* " 'Ich habe mich wohl gehütet, alle Patronen auf einmal zu verschießen.' Ernst Zermelo in Göttingen," *History and Philosophy of Logic* **11**, 19–58

— *1990b Hilbertprogramm und Kritische Philosophie. Das Göttinger Modell interdisziplinärer Zusammenarbeit zwischen Mathematik und Philosophie*, Vandenhoeck & Ruprecht, Göttingen (*Studien zur Wissenschafts-, Sozial- und Bildungsgeschichte der Mathematik*; 7).

— *1992* "Hilbert, Zermelo und die Institutionalisierung der mathematischen Logik in Deutschland," *Berichte zur Wissenschaftsgeschichte* **15**, 27–38.

— *1993* "Kurt Grelling und der Logische Empirismus," in R. Haller/F. Stadler (eds.), *Wien – Berlin – Prag. Der Aufstieg der wissenschaftlichen Philosophie. Zentenarien Rudolf Carnap – Hans Reichenbach – Edgar Zilsel*, Hölder-

Pichler-Tempsky, Vienna (*Veröffentlichungen des Instituts Wiener Kreis*; 2), 362–385.

— *1994a* "Logic in Transition: The Logical Calculi of Hilbert (1905) and Zermelo (1908)," in *Logic and Philosophy of Science in Uppsala. Papers from the 9th International Congress of Logic, Methodology and Philosophy of Science*, ed. by Dag Prawitz and Dag Westerståhl, Kluwer, Dordrecht/Boston/London (*Synthese Library*; 236), 311–323.

— *1994b* "Von Nelson zu Reichenbach. Kurt Grelling in Göttingen und Berlin," in L. Danneberg/A. Kamlah/L. Schäfer (eds.), Hans Reichenbach und die Berliner Gruppe, Vieweg & Sohn, Braunschweig/Wiesbaden, 53–73.

— *1995* "The Genesis of Grelling's Paradox," in I. Max/W. Stelzner (eds.), *Logik und Mathematik. Frege-Kolloquium Jena 1993*, Walter de Gruyter, Berlin/New York (*Perspectives in Analytical Philosophy*; 5), 269–280.

PECKHAUS, Volker/KAHLE, Reinhard *2002* "Hilbert's Paradox," *Historia Mathematica* **29**, 157–175.

PURKERT, Walter/ILGAUDS, Hans Joachim *1985 Georg Cantor*, Leipzig (*Biographien hervorragender Naturwissenschaftler, Techniker und Mediziner*; 79).

RAMSEY, Frank Plumpton *1926* "The Foundations of Mathematics," *Proceedings of the London Mathematical Society* (2) **25**, 338–384.

RANG, Bernhard/THOMAS, W. *1981* "Zermelo's Discovery of the 'Russell Paradox'," *Historia Mathematica* **8**, 15–22.

RÜSTOW, Alexander *1910 Der Lügner. Theorie, Geschichte und Auflösung*, Teubner, Leipzig (Ph. D. thesis Erlangen 1908).

RUSSELL, Bertrand *1903 The Principles of Mathematics*, The University Press, Cambridge [as "Vol. I"]; Allen & Unwin, London, ²1937; paperback edition Routledge, London 1992.

SCHRÖDER, Ernst *1990 Vorlesungen über die Algebra der Logik (exakte Logik)*, vol. 1, B. G. Teubner: Leipzig, reprinted as vol. 1 of the second edition Chelsea: Bronx, N. Y. 1966.

VAN HEIJENOORT, Jean *1967 From Frege to Gödel. A Source Book in Mathematical Logic, 1879–1931*, Cambridge, Mass., Harvard University Press.

WEYL, Hermann *1918 Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis*, Veit & Comp., Leipzig.

ZERMELO, Ernst *1901* "Ueber die Addition transfiniter Cardinalzahlen," *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-physikalische Klasse aus dem Jahre 1901* (Göttingen), 34–38.

— *1908a* "Neuer Beweis für die Möglichkeit einer Wohlordnung," *Mathematische Annalen* **65**, 107–128. Translated in van Heijenoort *1967*, 183–198.

— *1908b* "Untersuchungen über die Grundlagen der Mengenlehre. I," *Mathematische Annalen* **65**, 261–281. Translated in van Heijenoort *1967*, 199–215.