The Mathematical Origins of 19th Century Algebra of Logic*

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1 Introduction

Most 19th century scholars would have agreed to the opinion that philosophers are responsible for research on logic. On the other hand, the history of late 19th century logic clearly indicates a very dynamic development instigated not by philosophers, but by mathematicians. A central outcome of this development was the emergence of what has been called the "new logic", "mathematical logic", "symbolic logic", or, from 1904 on, "logistics".[1] This new logic came from Great Britain, and was created by mathematicians in the second half of the 19th century, finally becoming a mathematical subdiscipline in the early 20th century.

Charles L. Dodgson, better known under his pen name Lewis Carroll (1832–1898), published two well-known books on logic, *The Game of Logic* of *1887* and *Symbolic Logic* of *1896* of which a fourth edition appeared already in 1897. These books were written "to be of *real* service to the young, and to be taken up, in High Schools and in private families, as a valuable addition of their stock of healthful mental recreations" (Carroll *1896*, xiv). They were meant "to *popularize* this fascinating subject," as Carroll wrote in the preface of the fourth edition of *Symbolic Logic* (*ibid.*). But, astonishingly enough, in both books there is no definition of the term "logic". Given the broad scope of these books the title "Symbolic Logic" of the second book should at least have been explained.

*The text is based (but elaborated and enlarged) on my paper "19th Century Logic Between Philosophy and Mathematics" (Peckhaus *1999*).

[1] Independently of each other, Gregorius Itelson, André Lalande and Louis Couturat suggested at the 2nd Congress of Philosophy at Geneva in 1904 to use the name "logistic" for, as Itelson said, the modern kind of traditional formal logic. The name should replace designations like "symbolic", "algorithmic", "mathematical logic", and "algebra of logic" which were used synonymously up to then (cf. Couturat *1904*, 1042).

Maybe the idea of symbolic logic was so widely spread at the end of the 19th century in Great Britain that Carroll regarded a definition as simply unnecessary. Some further observations support this thesis. They concern a remarkable interest by the general public in symbolic logic, after the death of the creator of the algebra of logic, George Boole, in 1864.

Recalling some standard 19th century definitions of logic as, e.g., the art and science of reasoning (Whately) or the doctrine giving the normative rules of correct reasoning (Herbart), it should not be forgotten that mathematical or symbolic logic was not set up from nothing. It arose from the old *philosophical* collective discipline logic. It is therefore obvious to assume that there was some relationship between the philosophical and the mathematical side of the development of logic, but standard presentations of the history of logic ignore this putative relationship; they sometimes even deny that there has been any development of philosophical logic at all, and that philosophical logic could therefore justly be ignored.

Take for instance William and Martha Kneale's programme in their eminent *The Development of Logic*. They wrote (*1962*, iii): "But our primary purpose has been to record the first appearances of these ideas which seem to us most important in the logic of our own day," and these are the ideas leading to mathematical logic. Another example is J. M. Bocheński's assessment of "modern classical logic" which he dated between the 16th and the 19th century. This period was for him non-creative. It can therefore justly be ignored in a problem history of logic (*1956*, 14). According to Bocheński classical logic was only a decadent form of this science, a dead period in its development (ibid., 20).

Authors advocating such opinions adhere to the predominant views of present-day logic, i. e. actual systems of mathematical or symbolic logic. As a consequence, they are not able to give reasons for the final divorce between philosophical and mathematical logic, because they ignore the seed from which mathematical logic has emerged. Following Bocheński's view Carl B. Boyer presented for instance the following periodization of the development of logic (Boyer *1968*, 633): "The history of logic may be divided, with some slight degree of oversimplification, into three stages: (1) Greek logic, (2) Scholastic logic, and (3) mathematical logic." Note Boyer's "slight degree of oversimplification" which enabled him to skip 400 years of logical development and ignore the fact that Kant's transcendental logic, Hegel's metaphysics and Mill's inductive logic were called "logic", as well.

This restriction of scope had a further consequence: The history of logic is written as if it had been the 19th century mathematicians' main motive for doing logic to create and develop a new scientific discipline as such, namely mathematical logic, dealing above all with problems arising in this discipline and solving these problems with the final aim of attaining a coherent theory.

But what, if logic was only a means to an end, a tool for solving non-logical problems? If this is considered, such non-logical problems have to be taken note of. One can assume that at least the initial motives of mathematicians working in logic were going beyond creating a new or further developing the traditional theory of logic. Under the presupposition that a mathematician is usually not really interested in devoting his professional work to the development of a philosophical subdiscipline, one can assume that theses motives have to be sought in the mathematician's own subject, namely in foundational, i. e., philosophical problems of mathematics.

Today historians have recognized that the emergence of the new logic was no isolated process. Its creation and development ran parallel to and was closely intertwined with the creation and development of modern abstract mathematics which emancipated itself from the traditional definition as a science which deals with quantities and geometrical forms and is therefore responsible for *imaginabilia*, i. e. intuitive objects. The *imaginabilia* are distinguished from *intelligibilia*, i. e. logical objects which have their origin in reason alone. These historians recognized that the history of the development of modern logic can only be told within the history of the development of mathematics because the new logic is not conceivable without the new mathematics. In recent research on the history of logic this intimate relation between logic and mathematics, especially its connection to foundational studies in mathematics has been taken into consideration. One may mention the present author's *Logik, Mathesis universalis und allgemeine Wissenschaft* (Peckhaus *1997*) dealing with the philosophical and mathematical contexts of the development of 19th century algebra of logic as at least partially unconscious realizations of the Leibnitian programme of a universal mathematics, José Ferreirós' history of set theory in which the deep relations between the history of abstract mathematics and that of modern logic (Ferreirós *1999*) are unfolded, and the masterpiece of this new direction, *The Search for Mathematical Roots, 1870–1940* (*2000a*) by Ivor Grattan-Guinness, who imbedded the whole bunch of different directions in logic into the development of foundational interests within mathematics. William Ewald's "Source Book in the Foundations of Mathematics" (Ewald, ed., *1996*) considers logical influences at least in passing, whereas the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* edited by Ivor Grattan-Guinness (*1994*) devotes an entire Part to "Logic, Set Theories and the Foundation of Mathematics" (vol. 1, pt. 5).

In the following the complex conditions for the emergence of 19th century symbolic logic will be discussed. The main scope will be on the mathematical motives leading to the interest in logic, the philosophical context will be dealt with only in passing. The main object of study will be the algebra of logic in its British and German versions. Special emphasis will be laid on the systems of George Boole (1815–1864) and above all of his German follower Ernst Schröder (1841–1902).

2 George Boole's Algebra of Logic

2.1 Philosophical Context

The development of the new logic started in 1847, completely independent of earlier anticipations, e.g. those by the German rationalistic universal genius Gottfried Wilhelm Leibniz (1646–1716) and his followers (cf. Peckhaus *1994a*; *1997*, ch. 5). In that year the British mathematician George Boole (1815–1864) published his pamphlet *The Mathematical Analysis of Logic* (*1847*).[2] Boole mentioned (*1847*, 1) that it was the struggle for priority concerning the quantification of the predicate between the Edinburgh philosopher William Hamilton (1788–1856) and the London mathematician Augustus De Morgan (1806–1871) which encouraged this study. Hence, he referred to a startling philosophical discussion which indicated a vivid interest in formal logic in Great Britain. This interest was, however, a new interest, just 20 years old. One can even say that neglect of formal logic could be regarded as a characteristic feature of British philosophy up to 1826 when Richard Whately (1787–1863) published his *Elements of Logic*.[3] In his preface Whately added an extensive report on the languishing research and education in formal logic in England. He complained (*1826*, xv) that only very few students of the University of Oxford became good logicians and that

> by far the greater part pass through the University without knowing any thing of all of it; I do not mean that they have not learned by rote a string of technical terms; but that they understand absolutely nothing whatever of the principles of the Science.

Thomas Lindsay, the translator of Friedrich Ueberweg's important *System der Logik und Geschichte der logischen Lehren* (*1857*, translation *1871*), was very critical of the scientific qualities of Whately's book, but he, nevertheless, emphasized its outstanding contribution for the renaissance of formal logic in Great Britain (Lindsay *1871*, 557):

> Before the appearance of this work, the study of the science had fallen into universal neglect. It was scarcely taught in the universities, and there was hardly a text-book of any value whatever to be put into the hands of the students.

[2]For a book-length biography see MacHale *1985*. See also contemporary obituaries and biographies like Harley *1866*, Neil *1865*, both reprinted. For a comprehensive presentation of Boole's logic in the context of British mathematics, cf. Grattan-Guinness *2000a*.

[3]Whately *1826*. Risse (*1973*) lists nine editions up to 1848 and 28 further printings to 1908. Van Evra (*1984*, 2) mentions 64 printings in the USA to 1913.

One year after the publication of Whately's book, George Bentham's *An Outline of a New System of Logic* appeared (*1827*) which was intended as a commentary to Whately. Bentham's book was critically discussed by William Hamilton in a review article published in the *Edinburgh Review* (*1833*). With the help of this review Hamilton founded his reputation as the "first logical name in Britain, it may be in the world."[4] Hamilton propagated a revival of the Aristotelian scholastic formal logic without, however, one-sidedly preferring the syllogism. His logical conception was focused on a revision of the standard forms by quantifying the predicates of judgements.[5] The controversy about priority arose, when De Morgan, in a lecture "On the Structure of the Syllogism" (De Morgan *1846*) given to the Cambridge Philosophical Society on 9th November 1846, also proposed the quantification of the predicates.[6] Neither had any priority, of course. The application of diagrammatic methods in syllogistic reasoning proposed, e. g., by the 18th century mathematicians and philosophers Leonard Euler, Gottfried Ploucquet, and Johann Heinrich Lambert, presupposed a quantification of the predicate.[7] The German psychologistic logician Friedrich Eduard Beneke (1798–1854) suggested to quantify the predicate in his books on logic published in *1839* and *1842*, the latter of which he sent to Hamilton (cf. Peckhaus*1997*, 191–193). In the context of this presentation it is irrelevant to give a final solution of the priority question. It is, however, important that a dispute of this extent arose at all. It indicates that there was a new interest in *research* on formal logic.

This interest represented only one side of the effects released by Whately's book. Another line of research stood in the direct tradition of Humean empiricism and the philosophy of inductive sciences: the inductive logic of John Stuart Mill (1806–1873), Alexander Bain (1818–1903) and others. Boole's logic was in clear opposition to inductive logic. It was Boole's follower William Stanley Jevons (1835–1882; cf. Jevons *1877–1878*) who made this opposition explicit.

As mentioned earlier, Boole referred to the controversy between Hamilton and De Morgan, but this influence should not be overemphasized. In his main work on the *Laws of Thought* (*1854*) Boole went back to the logic of Aristotle by quoting from the Greek original. This can be interpreted as indicating that the influence of the contemporary philosophical discussion was

[4]This opinion can be found in a letter of De Morgan's to Spalding of 26th June, 1857 (quoted in Heath *1966*, xii) which was, however, not sent. George Boole lists Hamilton among the "two greatest authorities in logic, modern and ancient" (*1847*, 81). The other authority is Aristotle. This reverence to Hamilton might not be without irony because of Hamilton's disregard of mathematics.

[5]Cf. Hamilton *1859–1866*, vol. 4 (*1866*), 287.

[6]For the priority struggle see Heath *1966*.

[7]For diagrammatic methods in logic see Gardner *1958*, Bernhard *2001*.

not as important as his own words might suggest. In writing a book on logic he was doing philosophy, and it was thus a matter of course that he related his results to the philosophical discussion of his time. This does not mean, of course, that his thoughts were mainly influenced by this discussion. In any case, Boole's early algebra of logic kept a close connection to traditional logic, in the formal part of which the theory of syllogism represented its core.[8] Traditional logic not only provided the topics to be dealt with by the "Calculus of Deductive Reasoning",[9] it also served as a yardstick for evaluating the power and the reliability of the new logic. Even in the unpublished manuscripts of a sequel of the *Laws of Thought* entitled "The Philosophy of Logic" he discussed Aristotelian logic at length (cf., e. g., Boole *1997*, 133–136), criticizing, however, that it is more a mnemonic art than a science of reasoning.[10]

2.2 The Mathematical Context in Great Britain

Of greater importance than the philosophical discussion on logic in Great Britain were mathematical influences. Most of the new logicians can be related to the so-called "Cambridge Network" (Cannon *1978*, 29–71), i. e. a movement which aimed at reforming British science and mathematics which started at Cambridge. One of the roots of this movement was the foundation of the Analytical Society in 1812 (cf. Enros *1983*) by Charles Babbage (1791–1871), George Peacock (1791–1858) and John Herschel (1792–1871). Joan L. Richards called this act a "convenient starting date for the nineteenth-century chapter of British mathematical development" (Richards *1988*, 13). One of the first achievements of the Analytical Society was a revision of the Cambridge Tripos by adopting the Leibnizian notation for the calculus and abandoning the customary Newtonian theory of fluxions: "the principles of pure D-ism in opposition to the Dot-age of the University" as Babbage wrote in his memoirs (Babbage *1864*, 29). It may be assumed that this successful movement triggered off by a change in notation might have stimulated a new or at least revived interest in operating with symbols. This new research on the calculus had parallels in innovative approaches to algebra which were motivated by the reception of Laplacian analysis.[11] In the first place the development of symbolical algebra has to be mentioned. It was codified by

[8]Cf., e. g., the section "On Expression and Interpretation" in Boole *1847*, 20–25, in which Boole gives his reading of the traditional theory of judgement. The section is followed by an application of his notation to the theory of conversion (*ibid.*, 26–30), and of syllogism (*ibid.*, 31–47).

[9]This is the subtitle of Boole's *Mathematical Analysis of Logic* (*1847*).

[10]For the influence of Aristotelian logic on Boole's philosophy of logic, cf. Nambiar *2000*.

[11]On the mathematical background of Boole's *Mathematical Analysis of Logic*, cf. Laita *1977*, Panteki *2000*.

George Peacock in his *Treatise on Algebra* (*1830*) and further propagated in his famous report for the British Association for the Advancement of Science (Peacock *1834*, especially 198–207). Peacock started by drawing a distinction between arithmetical and symbolical algebra, which was, however, still based on the common restrictive understanding of arithmetic as the doctrine of quantity. A generalization of Peacock's concept can be seen in Duncan F. Gregory's (1813–1844) "calculus of operations". Gregory was most interested in *operations* with symbols. He defined symbolical algebra as "the science which treats of the combination of operations defined not by their nature, that is by what they are or what they do, but by the laws of combinations to which they are subject" (*1840*, 208). In his much praised paper "On a General Method in Analysis" (*1844*), Boole made the calculus of operations the basic methodological tool for analysis. However in following Gregory, he went further, proposing more applications. He cited Gregory who wrote that a symbol is defined algebraically "when its laws of combination are given; and that a symbol represents a given operation when the laws of combination of the latter are the same as those of the former" (Gregory *1842*, 153–154). It is possible that a symbol for an arbitrary operation can be applied to the same operation (*ibid.*, 154). It is thus necessary to distinguish between arithmetical algebra and symbolical algebra which has to take into account symbolical, but non-arithmetical fields of application. As an example Gregory mentioned the symbols a and $+a$. They are isomorphic in arithmetic, but in geometry they need to be interpreted differently. a can refer to a point marked by a line whereas the combination of the signs $+$ and a additionally expresses the direction of the line. Therefore symbolical algebra has to distinguish between the symbols a and $+a$. Gregory deplored the fact that the unequivocity of notation did not prevail as a result of the persistence of mathematical practice. Clear notation was only advantageous, and Gregory thought that our minds would be "more free from prejudice, if we never used in the general science symbols to which definite meanings had been appropriated in the particular science" (*ibid.*, 158).

Boole adopted this criticism almost word for word. In his *Mathematical Analysis of Logic* he claimed that the reception of symbolic algebra and its principles was delayed by the fact that in most interpretations of mathematical symbols the idea of quantity was involved. He felt that these connotations of quantitative relationships were the result of the context of the emergence of mathematical symbolism, and not of a universal principle of mathematics (Boole *1847*, 3–4). Boole read the principle of the permanence of equivalent forms as a principle of independence from interpretation in an "algebra of symbols". In order to obtain further affirmation, he tried to free the principle from the idea of quantity by applying the algebra of symbols to another field, the field of logic. As far as logic is concerned this implied that only the principles of a "true Calculus" should be presupposed. This calculus is characterized as a "method resting upon the employment of Symbols, whose laws of combination are known and general, and whose results admit of a consistent interpretation" (*ibid.*, 4). He stressed (*ibid.*):

> It is upon the foundation of this general principle, that I purpose to establish the Calculus of Logic, and that I claim for it a place among the acknowledged forms of Mathematical Analysis, regardless that in its objects and in its instruments it must at present stand alone.

Boole expressed logical propositions in symbols whose laws of combination are based on the mental acts represented by them. Thus he attempted to establish a psychological foundation of logic, mediated, however, by language.[12] The central mental act in Boole's early logic is the act of election used for building classes. Man is able to separate objects from an arbitrary collection which belong to given classes, in order to distinguish them from others. The symbolic representation of these mental operations follows certain laws of combination which are similar to those of symbolic algebra. Logical theorems can thus be proven like mathematical theorems. Boole's opinion has of course consequences for the place of logic in philosophy: "On the principle of a true classification, we ought no longer to associate Logic and Metaphysics, but Logic and Mathematics" (*ibid.*, 13).

Although Boole's logical considerations became increasingly philosophical with time, aiming at the psychological and epistemological foundations of logic itself, his initial interest was not to reform logic but to reform mathematics. He wanted to establish an abstract view on mathematical operations without regard to the objects of these operations. When claiming "a place among the acknowledged forms of Mathematical Analysis" (*1847*, 4) for the calculus of logic, he didn't simply want to include logic in traditional mathematics. The superordinate discipline was a *new* mathematics. This is expressed in Boole's writing: "It is not of the essence of mathematics to be conversant with the ideas of number and quantity" (*1854*, 12).

2.3 Boole's Logical System

Boole's early logical system is based on mental operations, namely acts of selecting individuals from classes. In his notation 1 symbolizes the Universe, comprehending "every conceivable class of objects whether existing or not" (*1847*, 15). Capital letters stand for all members of a certain class. The small letters are introduced as follows (*ibid.*, 15):

> The symbol x operating upon any subject comprehending individuals or classes, shall be supposed to select from that subject all the Xs

[12] On Boole's "psychologism", cf. Bornet *1997*, Vasallo *2000*.

> which it contains. In like manner the symbol y, operating upon any subject, shall be supposed to select from it all individuals of the class Y which are comprised in it and so on.

Take X as the class of animals, then x might signify the selection of all sheep from these animals, which then can be regarded as a new class from which we select further objects, etc. This might be illustrated by the following example:

animals		sheep		horned sheep		black horned sheep
X		Y		Z		
$\downarrow$	$\nearrow$	$\downarrow$	$\nearrow$	$\downarrow$	$\nearrow$	
x		y		z		
sheep		horned		black		

This process represents a successive selection which leads to individuals common to the classes X, Y, and Z. xyz stands for animals that are sheep, horned and black. It can be regarded as the logical sum of some common marks or common aspects relevant for the selection. In his major work *An Investigation of the Laws of Thought* of 1854 Boole gave up this distinction between capital and small letters, thereby getting rid of the complicated consequences of this stipulation.

If the symbol 1 denotes the universe, and if the class X is determined by the symbol x, it is consequent that the class not-X has to be denoted by the symbol $1-x$ which forms the supplement to x, thus $x(1-x) = 1$. 0 symbolizes nothing or the empty class. Now one can consider Boole's interpretation of the universal-affirmative judgement. The universal-affirmative judgement "All Xs are Ys" is expressed by the equation $xy = x$ or, by simple arithmetical transformation, $x(1 - y) = 0$ (p. 22):

> As all the Xs which exist are found in the class Y, it is obvious that to select out of the Universe all Ys, and from these to select all Xs, is the same as to select at once from the Universe all Xs.

The universal-negative judgement "No Xs are Ys" asserts that there are no terms common in the classes X and Y. All individuals common would be represented by xy, but they form the empty class. The particular-affirmative judgement "Some Xs are Ys" says that there are some terms common to both classes which form the class V. They are expressed by the elective symbol v. The judgement is thus represented by $v = xy$. Boole furthermore considers to use $vx = vy$ with vx for "some X" and vy for "some Y", but observes "that this system does not express quite so much as the single equation [...]" (p. 22–3). The particular-negative judgement "Some Xs are not Ys" can be reached by simply replacing y in the last formula with $1 - y$.

Boole's elective symbols are compatible to the traditional theory of judgement. They blocked, however, the step towards modern quantification theory as present in the work of Gottlob Frege, but also in later systems of the algebra of logic like those of Ch. S. Peirce and Ernst Schröder.[13]

The basic relation in the Boolean calculus is equality. It is governed by three principles which are themselves derived from elective operations (cf. *ibid.*, 16–18):

1. The *Distributivity of Elections* (16–17):

   > [...] it is indifferent whether from of group of objects considered as a whole, we select the class X, or whether we divide the group into two parts, select the Xs from them separately, and then connect the results in one aggregate conception,

   in symbols:
   $$x(u+v) = xu + xv \ ,$$
   with $u+v$ representing the undivided group of objects, and u and v standing for its component parts.

2. The *Commutativity of Elections*: The order of elections is irrelevant:
   $$xy = yx \ .$$

3. The *Index Law*: The successive execution of the same elective act does not change the result of the election:
   $$x^n = x \text{ ,for } n \geq 2.$$

Boole stressed the importance of the Index Law which is not generally valid in arithmetic (only in the arithmetic of 0 and 1) and therefore peculiar for elective symbols. It allows to reduce complex formulas to forms easier capable of being interpreted.

In his *An Investigation of the Laws of Thought* (*1854*) Boole abandoned the Index Law and replaced it by the Law of Duality ("Boole's Law") $xx = x$, or $x^2 = x$.[14] His esteem for this law becomes evident in his claim "that the axiom of the metaphysicians which is termed the principle of contradiction [...], is a consequence of the fundamental law of thought whose expression is $x^2 = x$" (Boole *1854*, 49). Boole referred to the derivation

[13] For the development of quantification theory in the algebra of logic see Brady *2000*.

[14] The reason was that already the factorization of $x^3 = x$ leads to uninterpretable expressions. On Boole's *Laws of Thought* cf. Van Evra *1977*; on the differences between Boole's earlier and later logical theory cf. Grattan-Guinness *2000b*.

$$\begin{aligned} x^2 &= x \\ x - x^2 &= 0 \\ x(1-x) &= 0\ , \end{aligned}$$

the last formula saying that a class and its complement have no elements in common. Boole was heavily criticized for this "curious error" (Halsted *1878*, 86) of considering the Law of Contradiction a consequence of the Law of Duality, not the other way around (the derivation works, of course, also in the other direction).

Boole's revisions came along with a change in his attitude towards logic. Boole's early logic can be seen as an application of a new mathematical method to logic, thereby showing the efficacy of this method within the broad project of a universal mathematics, and so serving foundational goals in mathematics. This foundational aspect diminished in later work, successively being replaced by the idea of a reform of logic. Already in the paper "The Calculus of Logic" (Boole *1848*) Boole tried to show that his logical calculus is compatible with traditional philosophical logic. Reasoning is guided by the laws of thought. They are the central topic in Boole's *An Investigation of the Laws of Thoughts* claiming that "there is to a considerable extend an exact agreement in the laws by wich the two classes of operations are conducted" (*1854*, 6), comparing thereby the laws of thought and the laws of algebra. Logic, in Boole's understanding, was "a normative theory of the products of mental processes" (Grattan-Guinness *2000*, 51).

2.4 Reception of the New Logic

Although created by mathematicians, the new logic was widely ignored by fellow mathematicians. George Boole was respected by British mathematicians, but his ideas concerning an algebraic representation of the laws of thought received very little published reaction.[15] He shared this fate with Augustus De Morgan, the second major figure of symbolic logic at that time.[16] In 1864, Samuel Neil, the early chronicler of British mid 19th century logic, expressed his thoughts about the reasons for this negligible reception: "De Morgan is esteemed crotchety, and perhaps formalizes too much. Boole demands high mathematic culture to follow and to profit from" (*1864*, 161). One should add that the ones who had this culture were usually not interested in logic.

The situation changed after George Boole's death in 1864. In the following comments only some ideas concerning the reasons for this new interest are hinted at. In particular the rôles of William Stanley Jevons and Alexander

[15]On initial reactions cf. Grattan-Guinness *2000a*, 54–59.

[16]For a discussion of De Morgan's logic cf. Grattan-Guinness *2000a*, 25–37; Merrill *1990*.

Bain are stressed. These examples show that a broader reception of symbolic logic commenced only when its relevance for the philosophical discussion of the time came to the fore.

2.4.1 William Stanley Jevons

A broader international reception of Boole's logic began when William Stanley Jevons (1835–1882) made it the starting point for his influential *Principles of Science* of *1874*. He used his own version of the Boolean calculus introduced in his *Pure Logic* of 1864. Among his revisions were the introduction of a simple symbolical representation of negation and the definition of logical addition as inclusive "or", thereby creating Boolean Algebra (cf. Hailperin *1981*). He also changed the philosophy of symbolism (*1864*, 5):

> The forms of my system may, in fact, be reached by divesting his [Boole's] of a mathematical dress, which, to say the least, is not essential to it. The system being restored to its proper simplicity, it may be inferred, not that Logic is a part of Mathematics, as is almost implied in Professor Boole's writings, but that the Mathematics are rather derivatives of Logic. All the interesting analogies or samenesses of logical and mathematical reasoning which may be pointed out, are surely reversed by making Logic the dependent of Mathematics.

Jevons' interesting considerations on the relationship between mathematics and logic representing an early logicistic attitude will not be discussed here. Similar ideas can be found not only in Gottlob Frege's work, but also in that of Hermann Rudolf Lotze (1817–1881) and Ernst Schröder (1841–1902). Most important in the present context, is the fact that Jevons abandoned mathematical symbolism in logic, an attitude which was later taken up by John Venn (1834–1923). Jevons attempted to free logic from the semblance of being a special mathematical discipline. He used the symbolic notation only as a means of expressing general truths. Logic became a tool for studying science, a new language providing symbols and structures. The change in notation brought the new logic closer to the philosophical discourse of the time. The reconciliation was supported by the fact that Jevons formulated his *Principles of Science* as a rejoinder to John Stuart Mill's (1806–1873) *A System of Logic* of *1843*, at that time the dominating work on logic and the philosophy of science in Great Britain. Although Mill called his logic *A System of Logic Ratiocinative and Inductive*, the deductive parts played only a minor rôle, used only to show that all inferences, all proofs and the discovery of truths consisted of inductions and their interpretations. Mill claimed to have shown "that all our knowledge, not intuitive, comes to us exclusively from that source" (Mill *1843*, Bk. II, ch. I, § 1). Mill concluded that the question as to what induction is, is the most important question of the science of logic, "the question which includes all others." As a result the logic of induction covers by far the largest part of this work, a subject which we would today regard as belonging to the philosophy of science.

Jevons defined induction as a simple inverse application of deduction. He began a direct argument with Mill in a series of papers entitled "Mill's Philosophy Tested" (*1877/78*). This argument proved that symbolic logic could be of importance not only for mathematics, but also for philosophy.

Another effect of the attention caused by Jevons was that British algebra of logic was able to cross the Channel. In 1877, Louis Liard (1846–1917), at that time professor at the Faculté de lettres at Bordeaux and a friend of Jevons, published two papers on the logical systems of Jevons and Boole (Liard *1877a*, *1877b*). In *1878* he added a booklet entitled *Les logiciens anglais contemporaines* which had five editions until 1907, and was translated into German in *1880*. Although Hermann Ulrici (1806–1884) had published a first German review of Boole's *Laws of Thought* as early as *1855* (cf. Peckhaus *1995*), the knowledge of British symbolic logic was conveyed primarily by Alois Riehl (1844–1924), then professor at the University of Graz, in Austria. He published a widely read paper "Die englische Logik der Gegenwart" ("English contemporary logic") in *1877* which reported mainly Jevons' logic and utilized it in a current German controversy on the possibility of scientific philosophy.

2.4.2 Alexander Bain

Surprisingly good support for the reception of Boole's algebra of logic came from the philosophical opposition, namely from the Scottish philosopher Alexander Bain (1818–1903) who was an adherent of Mill's logic. Bain's *Logic*, first published in 1870, had two parts, the first on deduction and the second on induction. He made explicit that "Mr Mill's view of the relation of Deduction and Induction is fully adopted" (*1870*, I, iii). Obviously he shared the "[...] general conviction that the utility of the purely Formal Logic is but small; and that the rules of Induction should be exemplified even in the most limited course of logical discipline" (*ibid.*, v). The minor rôle of deduction showed up in Bain's definition "*Deduction* is the application or extension of Induction to *new cases*" (40).

Despite his reservations about deduction, Bain's *Logic* became important for the reception of symbolic logic because of a chapter of 30 pages entitled "Recent Additions to the Syllogism." In this chapter the contributions of William Hamilton, Augustus De Morgan and George Boole were introduced. Presumably many more people became acquainted with Boole's algebra of logic through Bain's report than through Boole's own writings. One example is Hugh MacColl (1837–1909), the pioneer of the calculus of propositions (statements) and of modal logic.[17] He created his ideas independently of Boole, eventually realizing the existence of the Boolean calculus by means of Bain's report. Even in the early parts of his series of papers "The Calculus of Equivalent Statements" he quoted from Bain's presentation when discussing Boole's logic (MacColl *1877/78*). In 1875 Bain's logic was translated into French, in 1878 into Polish. Tadeusz Batóg and Roman Murawski (*1996*) have shown that it was Bain's presentation which motivated the first Polish algebraist of logic, Stanisław Piątkiewicz (1848–?) to begin his research on symbolic logic.

3 Ernst Schröder's Algebra of Logic

3.1 Philosophical Background

The philosophical discussion on logic after Hegel's death in Germany was still determined by a Kantian influence.[18] In the preface to the second edition of his *Kritik der reinen Vernunft* of *1787*, Immanuel Kant (1723–1804) had written that logic had followed the safe course of a science since earliest times. For Kant this was evident because of the fact that logic had been prohibited from taking any step backwards from the time of Aristotle. But he regarded it as curious that logic hadn't taken a step forward either (B VIII). Thus, logic seemed to be closed and complete. Formal logic, in Kant's terminology the analytical part of general logic, did not play a prominent rôle in Kant's system of transcendental philosophy. In any case, it was a negative touchstone of truth, as he stressed (B 84). Georg Wilhelm Friedrich Hegel (1770–1831) went further in denying any relevance of formal logic for philosophy (Hegel *1812/13*, I, Introduction, XV–XVII). Referring to Kant, he maintained that from the fact that logic hadn't changed since Aristotle one should infer that it needs to be completely rebuilt (*ibid.*, XV). Hegel created a variant of logic as the foundational science of his philosophical system, defining it as "the science of the pure idea, i.e., the idea in the *abstract element of reasoning*" (*1830*, 27). Hegelian logic thus coincides with metaphysics (*ibid.*, 34).

This was the situation when after Hegel's death philosophical discussion on formal logic started again in Germany. This discussion on logic reform stood under the label of "the logical question", a term coined by the Neo-Aristotelian Adolf Trendelenburg (1802–1872). In 1842 he published a paper entitled "Zur Geschichte von Hegel's Logik und dialektischer Methode" with the subtitle "Die logische Frage in Hegel's Systeme". But what is the logical question according to Trendelenburg? He formulated this question explicitly towards the end of his article: "Is Hegel's dialectical method of pure reasoning

[17] On MacColl and his logic cf. Astroh/Read (eds.) *1998*.

[18] See for the following chs. 3 and 4 of Peckhaus *1997*, and Vilkko *2002*.

a scientific procedure?" (*1842*, 414). In answering this question in the negative, he provided the occasion of rethinking the status of formal logic within a theory of human knowledge without, however, proposing a return to the old (scholastic) formal logic. The term "the logical question" was subsequently used in a less specific way. Georg Leonard Rabus, the early chronicler of the discussion on logic reform, wrote, e. g., that the logical question emerged from doubts concerning the justification of formal logic (*1880*, 1).

Although this discussion was clearly *connected* to formal logic, the so-called reform did not *concern* formal logic. The reason was provided by the Neo-Kantian Wilhelm Windelband who wrote in a brilliant survey on 19th century logic (*1904*, 164):

> It is in the nature of things that in this enterprise [i.e. the reform of logic] the lower degree of fruitfulness and developability power was on the side of formal logic. Reflection on the rules of the correct progress of thinking, the technique of correct thinking, had indeed been brought to perfection by former philosophy, presupposing a naive world view. What Aristotle had created in a stroke of genius, was decorated with the finest filigree work in Antiquity and the Middle Ages: an art of proving and disproving which culminated in a theory of reasoning, and after this constructing the doctrines of judgements and concepts. Once one has accepted the foundations, the safely assembled building cannot be shaken: it can only be refined here and there and perhaps adapted to new scientific requirements.

Windelband was very critical of English mathematical logic. Its quantification of the predicate allows the correct presentation of extensions in judgements, but it "drops hopelessly" the vivid sense of all judgements, which tend to claim or deny a material relationship between subject or predicate. It is "a logic of the conference table", which cannot be used in the vivid life of science, a "logical sport" which has its merits only in exercising the final acumen (*ibid.*, 166–167).

The philosophical reform efforts concerned primarily two areas:

1. the problem of a foundation of logic itself, dealt with using psychological and physiological means, thereby leading to new discussion on the question of priority between logic and psychology, and to various forms of psychologism and anti-psychologism (cf. Rath *1994*, Kusch *1995*);

2. the problem of the applicability of logic which led to an increased interest in the methodological part of traditional logic. The reform of applied logic attempted to bring philosophy in touch with the stormy development of mathematics and sciences in that time.

Both reform procedures had a destructive effect on the shape of logic and philosophy. The struggle with psychologism led to the departure of psychology (especially in its new, experimental form) from the body of philosophy at the beginning of the 20th century. Psychology became a new, autonomous scientific discipline. The debate on methodology resulted in the creation of the philosophy of science which was separated from the body of logic. The philosopher's ignorance of the development of formal logic caused a third departure: Part of formal logic was taken from the domain of the competence of philosophy and incorporated into mathematics where it was instrumentalized for foundational tasks. This was the philosophical background before which the emergence of symbolic logic in Germany, especially the logical work of the German mathematician Ernst Schröder (1841–1902) has to be seen.

3.2 The Mathematical Context in Germany

3.2.1 Logic and Formal Algebra

The examination of the British situation in mathematics at the time when the new logic emerged showed that the creators of the new logic were basically interested in a reform of mathematics by establishing an abstract view of mathematics which focused not on mathematical objects like quantities, but on symbolic operations with arbitrary objects. The reform of logic was only secondary. These results can be transferred to the situation in Germany without any problem.

The most important representative of the German algebra of logic was the mathematician Ernst Schröder (1841–1902)[19] who was regarded as having completed the Boolean period in logic (cf. Bocheński *1956*, 314). In his first pamphlet on logic, *Der Operationskreis des Logikkalkuls* (*1877*), he presented a critical revision of Boole's logic of classes, stressing the idea of the duality between logical addition and logical multiplication introduced by William Stanley Jevons in 1864. In 1890 Schröder started the large project of his monumental *Vorlesungen über die Algebra der Logik* which remained unfinished although it increased to three volumes with four parts, of which one appeared only posthumously (*1890*, *1891*, *1895a*, *1905*). Contemporaries regarded the first volume alone as completing the algebra of logic (cf. Wernicke *1891*, 196). Nevertheless, Schröder's logical theory kept, like the one of Boole, close contact to the traditional shape of logic. The Introduction of the *Vorlesungen* is full of references to that time's philosophical discussion on logic. Schröder even referred to the psychologistic discussion on the foundation of logic, and

[19]On Schröder's biography see his autobiographical note, Schröder *1901*, which became the base of Eugen Lüroth's widely spread obituary, Lüroth *1903*. See also Peckhaus *1997*, 234–238.

never really freed his logical theory from the traditional division of logic into the theories of concept, judgement and inference.

Schröder's opinion concerning the question as to what end logic is to be studied (cf. Peckhaus *1991*, *1994b*) can be drawn from an autobiographical note (written in the third person), published in the year before his death. It contains his own survey of his scientific aims and results. Schröder divided his scientific production into three fields:

1. A number of papers dealing with some of the current problems of his science.

2. Studies concerned with creating an "absolute algebra," i. e., a general theory of connections. Schröder stressed that these studies represent his "very own object of research" of which only little was published at that time.

3. Work on the reform and development of logic.

Schröder wrote (*1901*) that his aim was

> to design logic as a calculating discipline, especially to give access to the exact handling of relative concepts, and, from then on, by emancipation from the routine claims of spoken language, to withdraw any fertile soil from "cliché" in the field of philosophy as well. This should prepare the ground for a scientific universal language that, widely differing from linguistic efforts like Volapük [a universal language like Esperanto, very popular in Germany at that time], looks more like a sign language than like a sound language.

Schröder's own division of his fields of research shows that he didn't consider himself a logician: His "very own object of research" was "absolute algebra," which was similar to modern abstract or universal algebra in respect to its basic problems and fundamental assumptions. What was the connection between logic and algebra in Schröder's research? From the passages quoted one could assume that they belong to two separate fields of research, but this is not the case. They were intertwined in the framework of his heuristic idea of a general science. In his autobiographical note he stressed:

> The disposition for schematizing, and the aspiration to condense practice to theory advised Schröder to prepare physics by perfecting mathematics. This required deepening—as of mechanics and geometry—above all of arithmetic, and subsequently he became by the time aware of the necessity for a reform of the source of all these disciplines, logic.

Schröder's universal claim becomes obvious. His scientific efforts served for providing the requirements to found physics as the science of material nature by "deepening the foundations," to quote a famous metaphor later used by David Hilbert (*1918*, 407) in order to illustrate the objectives of his axiomatic programme. Schröder regarded the formal part of logic that can be formed as a "calculating logic," using a symbolical notation, as a *model* of formal algebra that is called "absolute" in its last state of development.

But what is "formal algebra"? The theory of formal algebra "in the narrowest sense of the word" includes "those investigations on the laws of algebraic operations [...] that refer to nothing but general numbers in an unlimited number field without making any presuppositions concerning its nature" (*1873*, 233). Formal algebra therefore prepares "studies on the most varied number systems and calculating operations that might be invented for particular purposes" (*ibid.*).

It has to be stressed that Schröder wrote his early considerations on formal algebra and logic without any knowledge of the results of his British predecessors. His sources were the textbooks of Martin Ohm, Hermann Günther Graßmann, Hermann Hankel and Robert Graßmann. These sources show that Schröder was a representative of the tradition of German combinatorial algebra and algebraic analysis (cf. Peckhaus *1997*, ch. 6).

3.2.2 Combinatorial Analysis

Schröder developed the programmatic foundations of the absolute algebra in his textbook *Lehrbuch der Arithmetik und Algebra* (*1873*) and the school programme pamphlet *Über die formalen Elemente der absoluten Algebra* (*1874*). Among the sources mentioned in the textbook, Martin Ohm's (1792–1872) *Versuch eines vollkommen consequenten Systems der Mathematik* (*1822*) occurs that stood in the German tradition of the of algebraical and combinatorial analysis beginning with the work of Carl Friedrich Hindenburg (1741–1808) and his school (cf. Jahnke *1990*, 161–322).

Martin Ohm (cf. Bekemeier *1987*) aimed at completing Euclid's geometrical programme for all of mathematics (Ohm *1853*, V). He distinguished between number (or "undesignated number") and quantity (or "designated number") regarding the first one as the higher concept. The features of the calculi of arithmetic, algebra, analysis, etc. are not seen as features of quantities, but of operations, i. e. mental activities (*1853*, VI–VII). This operational view can also be found in the work of Hermann Günther Graßmann who also stood in the Hindenburg tradition.

3.2.3 General Theory of Forms

Hermann Günther Graßmann's *Lineale Ausdehnungslehre* (*1844*) was of decisive influence on Schröder, especially Graßmann's "general theory of forms" ("allgemeine Formenlehre") opening this pioneering study in vector algebra and vector analysis. The general theory of forms was popularized by Hermann Hankel's *Theorie der complexen Zahlensysteme* (*1867*).

Graßmann defined the general theory of forms as "the series of truths that is related to all branches of mathematics in the same way, and that therefore only presupposes the general concepts of equality and difference, connection and division" (*1844*, 1). Equality is taken as substitutivity in every context. Graßmann choses $\frown$ as general connecting sign. The result of the connection of two elements a and b is expressed by the term $(a \frown b)$. Using the common rules for brackets we get for three elements $((a \frown b) \frown c) = a \frown b \frown c$ (§ 2). Graßmann restricted his considerations to "simple connections", i. e. associative and commutative connections (§ 4). These connecting operations are synthetic. The reverse operations are called resolving or analytic connections. $a \smile b$ stands for the form which results in a if it is synthetically connected with b: $a \smile b \frown b = a$ (§ 5). Graßmann introduced furthermore forms in which more than one synthetic operation occur. If the second connection is symbolized with $\widehat{\frown}$ and if there holds distributivity between the synthetic operations, then the equation $(a \frown b) \widehat{\frown} c = (a \widehat{\frown} c) \frown (b \widehat{\frown} c)$ is valid. Graßmann called the second connection a connection on a higher level (§ 9), a terminology which might have influenced Schröder's later "Operationsstufen", i. e. "levels of operations".

Whereas Graßmann applied the general theory of forms in the domain of extensive quantities, especially directed lines, i. e. vectors, Hermann Hankel later used it to erect on its base his system of hypercomplex numbers (Hankel *1867*). If $\lambda(a, b)$ is a general connection of objects a, b leading to a new object c, i. e. $\lambda(a, b) = c$, there is a connection Θ which, applied to c and b leads again to a, i. e., $\Theta(c, b) = a$ or $\Theta\{\lambda(a, b), b\} = a$. Hankel called the operation θ "thetic" and its reverse λ "lytic". The commutativity of these operations is not presupposed.

3.2.4 "Wissenschaftslehre" and Logic

Hermann Günther Graßmann had already announced that his *Lineale Ausdehnungslehre* should be part of a comprehensive reorganization of the system of sciences. His brother Robert Graßmann (1815–1901) attempted to realize this programme in a couple of writings published under the series title *Wissenschaftslehre oder Philosophie*. In its parts on logic and mathematics he anticipated modern lattice theory. He furthermore formulated a logical calculus being in parts similar to that of Boole. His logical theory was obviously independent from the contemporary German philosophical discussion on logic, and he was also not aware of his British precursors.[20] Graßmann

[20] On Graßmann's logic and his anticipations of lattice theory cf. Mehrtens *1979*.

wrote about the aims of his logic or theory of reasoning ("Denklehre") that it

> should teach us strictly scientific reasoning which is equally valid for all men of any people, any language, equally proving and rigorous. It has therefore to relieve itself from the barriers of a certain language and to treat the forms of reasoning, becoming, thus, a *theory of forms* or *mathematics*.

Graßmann tried to realize this programme in his *Formenlehre oder Mathematik*, published in six brochures consisting of an introduction (*1872a*), a general part on "Grösenlehre" (*1872b*) understood as "science of tieing quantities" and the special parts "Begriffslehre oder Logik" (theory of concepts or logic), "Bindelehre oder Combinationslehre" (theory of binding or combinatorics), "Zahlenlehre oder Arithmetik" (theory of numbers or arithmetic) and "Ausenlehre oder Ausdehnungslehre" (theory of the exterior or Ausdehnungslehre).

In the general theory of quantities Graßmann introduced the letters a, b, c, ... as syntactical signs for arbitrary quantities. The letter e represents special quantities: elements, or in Graßmann's strange terminology "Stifte" (pins), i. e. quantities which cannot be derived from other quantities by tieing. Besides brackets which indicate the order of the tieing operation he introduces the equality sign $=$, the inequality sign $\angle$ and a general sign for a tie $\circ$. Among special ties he investigates joining or addition ("Fügung oder Addition") ("+") and weaving or multiplication ("Webung oder Multiplikation") ("·"). These ties can occur either as interior ties, if $e \circ e = e$, or as exterior exterior ties, if $e \circ e \angle e$.

The special parts of the theory of quantities are distinguished with the help of the combinatorically possible results of tieing a pin to itself. The first part, "the most simple and, at the same time, the most interior", as Graßmann called it, is the theory of concepts or logic in which interior joining $e + e = e$ and inner weaving $ee = e$ hold. In the theory of binding or combinatorics interior joining $e + e = e$ and exterior weaving $ee \angle e$ hold; in the theory of numbers or arithmetic exterior joining $e + e \angle e$ and interior weaving $ee = e$ hold, or $1 \times 1 = 1$ and $1 \times e = e$. Finally, in the theory of the exterior or Ausdehnungslehre, the "most complicated and most exterior" part of the theory of forms, exterior joining $e + e \angle e$ and exterior weaving $ee \angle e$ hold (*1872a*, 12–13).

Graßmann, thus, formulated Boole's "Law of Duality" using his interior weaving $ee = e$, but he went beyond Boole in allowing interior joining $e+e = e$, so coming close to Jevons' system of 1864.

In the theory of concepts or logic Graßmann started with interpreting the syntactical elements, which had already been introduced in a general way.

Now, everything that can be a definite object of reasoning is called "quantity". In this new interpretation, pins are initially set quantities not being derived from other quantities by tieing. Equality is interpreted as substitutivity without value change, inequality as impossibility of such a substitution. Joining is read as "and", standing for adjunction or the logical "or". Weaving is read as "times", i. e. conjunction or the logical "and". Graßmann introduced the signs $<$ and $>$ to express sub- and superordination of concepts. The sign $\leqq$ expresses, that a concept equals or that it is subordinated another concept. This is exactly the sense of Schröder's later basic connecting relation of subsumption or inclusion. In the theory of concepts Graßmann expressed this relation in a shorter way with the help of the angle sign $\angle$. The sign T stands for the All or the totality, the sum of all pins. The following laws hold: $a + \mathrm{T} = \mathrm{T}$ and $a\mathrm{T} = a$. 0 is interpreted as "the lowest concept, which is subordinate to all concepts." Its laws are $a + 0 = a$ and $a \cdot 0 = 0$. Finally Graßmann introduced the "not" ("Nicht") or negation as complement with the laws $a + \bar{a} = \mathrm{T}$ and $a \cdot \bar{a} = 0$.

3.3 Schröder's Algebra of Logic

3.3.1 Schröder's Way to Logic

In his work on the formal elements of absolute algebra (*1874*) Schröder investigated operations in a manifold, called domain of numbers ("Zahlengebiet"). "Number" is, however, used as a general concept. Examples for numbers are "proper names, concepts, judgements, algorithms, numbers [of arithmetic], symbols for quantities and operations, points, systems of points, or any geometrical object, quantities of substances, etc." (Schröder *1874*, 3). Logic is, thus, a possible interpretation of the structure dealt with in absolute algebra. Schröder assumed that there are operations with the help of which two objects from a given manifold can be connected to yield a third that also belongs to that manifold (*ibid.*, 4). He chooses from the set of possible operations the non-commutative "symbolic multiplication"

$$c = a \;.\; b = ab$$

with two inverse operations

$$\begin{array}{ll} \text{measuring ("Messung")} & b \;.\; (a : b) = a \;, \\ \text{and division ("Teilung")} & \frac{a}{b} \;.\; b = a \;. \end{array}$$

Schröder called a direct operation together with its inverses "level of operations" ("Operationsstufe"). And again Schröder realized that "the logical addition of concepts (or individuals)" follows the laws of multiplication of real numbers.

But there is still another association with logic. In his *Lehrbuch*, Schröder speculated about the relation between an "ambiguous expression" like $\sqrt{a}$ and its possible values. He determined five logical relations, introducing his subsumtion relations. Be A an expression that can have different values $a, a', a'', \ldots$. Then the following relations hold (Schröder *1873*, 27–29):

$$\textit{Superordination} \quad A \succcurlyeq \begin{cases} a \\ a' \\ a'' \\ \vdots \end{cases} \quad ,$$

Examples: metal $\succcurlyeq$ silver; $\sqrt{9} \succcurlyeq -3$.

$$\textit{Subordination} \quad \left.\begin{matrix} a \\ a' \\ a'' \\ \vdots \end{matrix}\right\} \preccurlyeq A \quad ,$$

Examples: gold $\preccurlyeq$ metal; $3 \preccurlyeq \sqrt{9}$.

$$\textit{Coordination} \quad a \asymp a' \asymp a'' \asymp \ldots \quad ,$$

Examples: gold $\asymp$ silber [in respect to the general concept "metal"] or $3 \asymp -3$ [in respect to the general concept $\sqrt{9}$].

$$\textit{Equality} \quad A = B$$

means that the concepts A and B are identical in intension and extension.

$$\textit{Correlation} \quad A(=)B$$

means that the concepts A and B agree in at least one value.

Schröder recognized that if he would now introduce *negation*, he would have created a complete terminology which allows to express all relations between concepts (in respect to their extension) with short formulas which can harmonically be embedded into the schema of the apparatus of the mathematical sign language (*ibid.*, 29).

Schröder wrote his logical considerations of the introduction of the *Lehrbuch* without having seen any work of logic in which symbolical methods had been applied. It was while completing a later sheet of his book that he came across Robert Graßmann's *Formenlehre oder Mathematik* (*1872a*). He felt urged to insert a comprehensive footnote running over three pages for hinting at this book (Schröder *1873*, footnote, pp. 145–147). There he reported that Graßmann used the sign + for the "collective comprehension", "really regarding it as an *addition*—one could say a 'logical' addition—that has besides the features of common (numerical) addition the basic feature $a + a = a$." He wrote that he was most interested in the rôle the author had assigned to multiplication regarded as the product of two concepts which unite the marks being common to both concepts.

In the *Programmschrift* of 1874 Schröder also gave credit to Robert Graßmann, but mentioned that he had recently found out that the laws of the logical operations had already been developed before Graßmann "in a classical work" by George Boole (Schröder *1874*, 7).

3.3.2 Logic as a Model of Absolute Algebra

In 1877 Schröder published his *Operationskreis des Logikkalkuls* in which he developed the logic of Boole's *Laws of Thought* stressing the duality of the logical operations of addition and multiplication. An "Operationskreis" (circle of operations) is constituted by more than one direct operation together with their inverses. The "logical calculus" is the set of formulas which can be produced in this circle of operations. Schröder called it a characteristic mark of "*mathematical logic* or the *logical calculus*" that these derivations and inferences can be done in form of calculations, namely in the first part of logic as calculation with concepts leading to statements about the objects themselves, i. e., categorical judgements, or, in Boole's terminology, "primary propositions". In its second part the logical calculus deals with statements about judgements as in conditional sentences, hypothetical or disjunctive judgements, or Boole's secondary propositions. In this booklet Schröder simplified Boole's calculus stressing, as mentioned, the duality between logical addition and logical multiplication, and, thus, the algebraic identity of the structures of these operations.

Schöder developed his logic in a systematic way in the *Vorlesungen über die Algebra der Logik* (*1890–1905*). Again he separated logic from its structure. The structures are developed and interpreted in several fields, beginning from the most general field of "domains" ("Gebiete") of manifolds of elements (which are not necessarily to be individuated), then classes (with and without negation), and finally proprositions (vol. 2, 1891). The basic operation in the calculi of domains and classes is subsumption, i. e., identity or inclusion. Schröder presupposes two principles, *Reflexivity* $a \leqslant a$, and *Transitivity* "If $a \leqslant b$ and at the same time $b \leqslant c$, then $a \leqslant c$". Then he defines "identical zero" ("nothing") and "identical one" ("all"), "identical multiplication" and "identical addition", and finally negation. In the sections dealing with statements without negation he proves one direction of the distributivity law for logical addition and logical multiplication, but shows that the other side cannot be proved, he rather shows its independence by formulating a model in which it does not hold, the so-called "*logical calculus with groups*, e. g. functional equations, algorithms or calculi". He thereby found the first example of a non-distributive lattice.[21]

Schröder devoted the second volume of the *Vorlesungen* to the calculus of propositions. The step from the calculus of classes to the calculus of propositions is taken with the help of an alteration of the basic interpretation of the formulas used. Whereas the calculus of classes was bound to a spatial interpretation especially in terms of the part–whole relation, Schröder used in the calculus of propositions a temporal interpretation taking up an idea of George Boole from his *Laws of Thought* (*1854*, 164–165). This may be illustrated regarding subsumtion as the basic connecting relation. In the calculus of classes $a \preccurlyeq b$ means that the class a is part of or equal to the class b. In the calculus of propositions this formula may be interpreted in the following way (Schröder *1891*, § 28, p. 13):

> In the time during which a is true is completely contained in the time during which b is true, i. e., *whenever* [...] *a is valid b is valid* as well. In short, we will often say: "*If* a is valid, then b is valid," "*a entails b*" [...], "from a follows b."

Schröder then introduces two new logical symbols, the "sign of products" $\prod$, and the "sign of sums" $\sum$. He uses $\prod_x$ to express that propositions referring to a domain x are valid for any domain x in the basic manifold 1, and $\sum_x$ to say that the proposition is not necessarily valid for all, but for a certain domain x, or for several certain domains x of our manifold 1, i. e., for at least one x (Schröder *1891*, § 29, 26–27).

For Schröder the use of $\sum$ and $\prod$ in logic is perfectly analogues to arithmetic. The existential quantifier and the universal quantifier are therefore interpreted as possibly indefinite logical addition or disjunction and logical multiplication or conjunction respectively. This is expressed by the following definition which also shows the duality of $\sum$ and $\prod$ (Schröder *1891*, § 30, 35).

$$\sum_{\lambda=1}^{\lambda=n} a_\lambda = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \quad \Big| \quad \prod_{\lambda=1}^{\lambda=n} a_\lambda = a_1 a_2 a_3 \cdot \ \ldots \ \cdot a_{n-1} a_n \ .$$

With this Schröder had all requirements at hand for modern quantification theory which he took, however, not from Frege, but from the conceptions

[21] Cf. Schröder *1890*, 280. On Ch. S. Peirce's claim to have proved the second form as well (Peirce *1880*, 33) cf. Houser *1991*. On Schröder's proof cf. Peckhaus *1994a*, 359–374; Mehrtens *1979*, 51–56.

as developed by Charles S. Peirce (1836–1914) and his school, especially by Oscar Howard Mitchell (1851–1889).[22]

3.3.3 Logic of Relatives

Schröder devoted the third volume of the *Vorlesungen* to the "Algebra and Logic of Relatives" of which only a first part dealing with the *algebra* of relatives could be published (Schröder *1895a*). The algebra and logic of relatives should serve as an organon for absolute algebra in the sense of pasigraphy, or general script, that could be used to describe most different objects as models of algebraic structures.

Schröder never claimed any priority for this part of his logic, but always conceded that it was an elaboration of Charles S. Peirce's work on relatives (cf. Schröder *1905*, XXIV).

He illustrated the power of this new tool by applying it to several mathematical topics, such as open problems of G. Cantor's set theory (e. g. Schröder *1898*), thereby proving (not entirely correctly) Cantor's proposition about the equivalence of sets ("Schröder-Bernstein Theorem"). In translating Richard Dedekind's theory of chains into the language of the algebra of relatives he even proclaimed the "*final* goal: to come to a strictly logical *definition* of the *relative* concept '*number of –*' ['*Anzahl von –*'] from which all propositions referring to this concept can be deduced purely deductively" (Schröder *1895a*, 349–350). So Schröder's system comes close, at least in its objectives, to Frege's logicism, although it is commonly regarded as an antipode.

4 Conclusions

Like the British tradition, but independent of it, the German algebra of logic was connected to new trends in algebra. It differed from its British counterpart in its combinatorial approach. In both traditions, algebra of logic was invented within the enterprise to reform basic notions of mathematics which led to the emergence of structural abstract mathematics. The algebraists wanted to design algebra as "pan-mathematics", i. e. as a general discipline embracing all mathematical disciplines as special cases. The independent attempts in Great Britain and Germany were combined when Schröder learned about the existence of Boole's logic in late 1873, early 1874. Finally he enriched the Boolean class logic by adopting Charles S. Peirce's theory of quantification and adding a logic of relatives according to the model of Peirce and De Morgan.

[22] Cf. Mitchell *1883*, Peirce *1885*. On the development of modern quantification theory in the algebra of logic cf. Brady *2000*.

The main interest of the new logicians was to utilize logic for mathematical and scientific purposes, and it was only in a second step, but nevertheless an indispensable consequence of the attempted applications, that the reform of logic came into the view. What has been said of the representatives of the algebra of logic also holds for the proponents of competing logical systems such as Gottlob Frege or Giuseppe Peano. They wanted to use logic in their quest for mathematical rigor, something questioned by the stormy development in mathematics.

For quite a while the algebra of logic remained the first choice for logical research. Author like Alfred North Whitehead (1841–1947), and even David Hilbert and his collaborators in the early foundational programme (cf. Peckhaus *1994c*) built on this direction of logic whereas Frege's mathematical logic was widely ignored. The situation changed only after the publication of A. N. Whitehead's and B. Russell's *Principia Mathematica* (*1910–1913*). But even then important work was done in the algebraic tradition as the contributions of Clarence Irving Lewis (1883–1964), Leopold Löwenheim (1878–1957), Thoralf Skolem (1887–1963), and Alfred Tarski (1901–1983) prove.

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